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Generation of random stress tensors

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ABSTRACT

To correctly incorporate stress variability in the increasingly widespread application of probabilistic-related rock mechanics analyses, a robust approach for random stress tensor generation is essential. However, currently, the customary scalar/vector approaches to the generation of random stress tensors, which violate the tensorial nature of stress, together with other existing quasi-tensorial applications that consider the tensor components as statistically independent variables, may yield biased results. Here, we propose a multivariate random vector generation approach for generating random stress tensor components that is based on tensorial techniques and which incorporates inter-component correlation. Differences between the proposed fully tensorial and existing quasi-tensorial approaches are demonstrated by examining the distributions of the tensors generated using both approaches, and the efficacy and transformational consistency of the proposed fully tensorial approach are investigated by generating random tensors in different coordinate systems. Our results suggest application of the existing quasi-tensorial approach (which ignores covariance) leads to greater scatter in generated tensors than does application of the proposed fully tensorial approach (which includes covariance). Additionally, the transformational consistency of the proposed fully tensorial approach allows generation of random tensors in any convenient coordinate system, while the existing quasi-tensorial approach only permits generation of random tensors in a particular coordinate system. The proposed fully tensorial approach provides a method that will assist with probabilistic-related analyses of rock engineering structures.

1. Introduction

In situ stress is an important parameter for a wide range of endeavours in rock mechanics, including rock engineering design, hydraulic fracturing analysis, rock mass permeability and evaluation of earthquake potential.[1–5] Because of the inherent complexity of fractured rock masses in terms of varying rock properties, the presence of discontinuities and unclear boundary conditions,[4] stress in rock often displays significant variability.[6] With the increasingly widespread application of probabilistic or reliability-based analyses in rock mechanics, incorporating stress variability in these analyses is becoming a necessity.[7–12] A robust approach for random stress tensor generation – i.e. one that is faithful to the tensorial nature of stress – is essential for such work. Here, and particularly to assist probabilistic-related analyses in rock mechanics that need to consider the inherent variability of *in situ* stress, we present a fully tensorial technique for generating random stress tensors.

Currently in rock mechanics, stress magnitude and orientation are customarily processed separately (e.g. Fig. 1). This processing effectively decomposes the second order stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, to which classical statistics[13] and directional statistics[14], respectively, are applied.[6,7,15–26] Following this, probabilistic analyses are generally implemented by drawing random variates separately from the statistical distributions of both principal stress magnitude and orientation.[7] These customary scalar/vector approaches violate the tensorial nature of stress and may yield biased results.[27–30] In particular, orthogonality of the randomly generated principal stresses is not guaranteed.

Rather than analysing principal stress magnitude and orientation separately, and in order to remain faithful to the tensorial nature of stress, stress analysis should be conducted on the basis of tensor components obtained in a common Cartesian coordinate system. Several researchers have followed this technique in random stress tensor generation,[31–33] with the random tensors being based on the mean and variance of each tensor component relative to a common coordinate system. However, this existing quasi-tensorial approach considers the tensor components as statistically independent variables, and ignores any correlation between them. The result is, to date there seems to have been no mathematically rigorous proposal from the rock mechanics community for random stress tensor generation.

Stress tensors, which are 2×2 or 3×3 symmetric matrices, together

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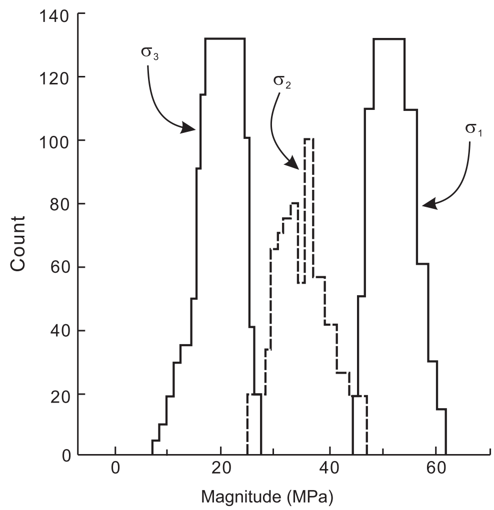
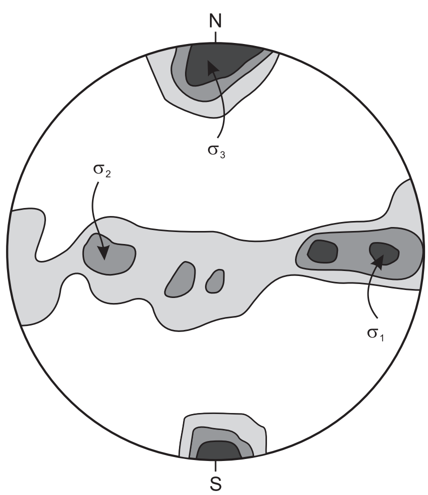


(a) Distribution of principal stress magnitudes



(b) Contouring of principal stress orientations

Fig. 1. Customary analyses of stress examine principal stress magnitude and orientation separately using classical statistics and directional statistics, respectively (after Brady & Brown[26]).

with other matrix-valued quantities, play a pivotal role in many subjects such as solid mechanics, physics, earth science, medical imaging and economics.[34] To explicitly account for the inherent variability of such matrix-valued quantities, matrix variate statistics – as a generalisation of multivariate statistics – has been developed,[34] and we have previously demonstrated that this statistics is appropriate for stress variability analysis.[35] Matrix variate statistics and multivariate statistics are often used interchangeably by statisticians,[36–38] and it has been demonstrated that matrix variate analysis of stress tensors and multivariate analysis of their distinct components are statistically equivalent.[34,35,39] Thus, from the viewpoint of random tensor generation, instead of generating the whole stress tensor, it is correct and more convenient to form a stress tensor by generating the distinct tensor components in a multivariate manner. Here, we use "distinct tensor components", rather than the customary "independent tensor components", and the reason for this is discussed later.

In the present paper, in order to propose a robust method for generating random stress tensors, we first examine related work in rock mechanics. We discuss the deficiency of processing principal stress magnitude and orientation separately, and the inappropriateness of the customary scalar/vector random stress generation approach, and examine the applicability of the existing quasi-tensorial applications found in the literature. Then, using a multivariate normal distribution model of the distinct tensor components as an example, we present a multivariate random vector generation approach for generating random stress tensor components that incorporates inter-component correlation. We illustrate the difference between the existing quasi-tensorial and new approaches, and by analysing actual stress data we demonstrate the efficacy of the proposed fully tensorial approach by examining the distributions of the tensors generated using both approaches in terms of tensor components and principal stresses. Finally, the transformational consistency of the proposed fully tensorial approach is illustrated by generating random tensors in different coordinate systems.

2. Related work

2.1. Deficiency of the customary scalar/vector approach

As noted above, the customary scalar/vector approach employed in rock mechanics of processing principal stress magnitude and orientation separately may yield unreasonable results. Here, we re-present the succinct and clear example presented in Dyke et al.[29] to emphasise this.

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the two stress states

$$\mathbf{S}_1 = \begin{bmatrix} 18 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } \mathbf{S}_2 = \begin{bmatrix} 10 & 0 \\ 0 & 18 \end{bmatrix}, \tag{1}$$

referred to a common Cartesian coordinate system. These stress states are also represented in Fig. 2a by ellipses whose semi-axes denote the magnitude and orientation of their principal values. $\mathbf{S}_1$ and $\mathbf{S}_2$ clearly possess identical principal stress magnitudes, but different principal stress orientations. If we separately determine the principal stress mean magnitudes and mean orientations, the result is a mean represented by ellipse A shown in Fig. 2b. However, when the tensorial approach that averages the corresponding tensor components[40] is applied, the mean symbolised by ellipse B results (Fig. 2b). If we apply the principles of solid mechanics and consider $\mathbf{S}_1$ and $\mathbf{S}_2$ as perturbations from some mean state, then the tensorial mean of

$$\bar{\mathbf{S}} = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} \tag{2}$$

is clearly correct. In essence, the customary scalar/vector approach is deficient in that it averages stress states (i.e. principal stresses) that are referred to their own, potentially unique, coordinate systems. This both violates the tensorial nature of stress and erroneously applies statistical tools to process data that are referred to different geometrical bases.

Although this example concerns only the case of calculating the mean of two stresses, this fundamental reasoning applies also to the case of additional stress tensors, as well as to statistics such as dispersion calculation, distribution characterisation and generation of random stress tensors. Thus, the conclusion to be drawn is that statistical and probabilistic applications based on separate processing of principal stress magnitude and orientation will be incorrect and may yield unreasonable results. Since the generation of random stress tensors depends on the underlying statistical model, randomly generating stress magnitude and orientation separately will be inappropriate. Instead, random stress tensors should be generated using tensorial approaches that generate random tensor components referred to a common Cartesian coordinate system, and, as shown below, some reports of this exist in the literature.

2.2. Existing quasi-tensorial random tensor generation approaches

A survey of the rock mechanics literature reveals the existence of a tensorial approach to random tensor generation that is based on the means and variances of the distinct tensor components of measured *in situ* stress data, and which generates random tensors in the coordinate system that aligns with the direction of the principal components of the

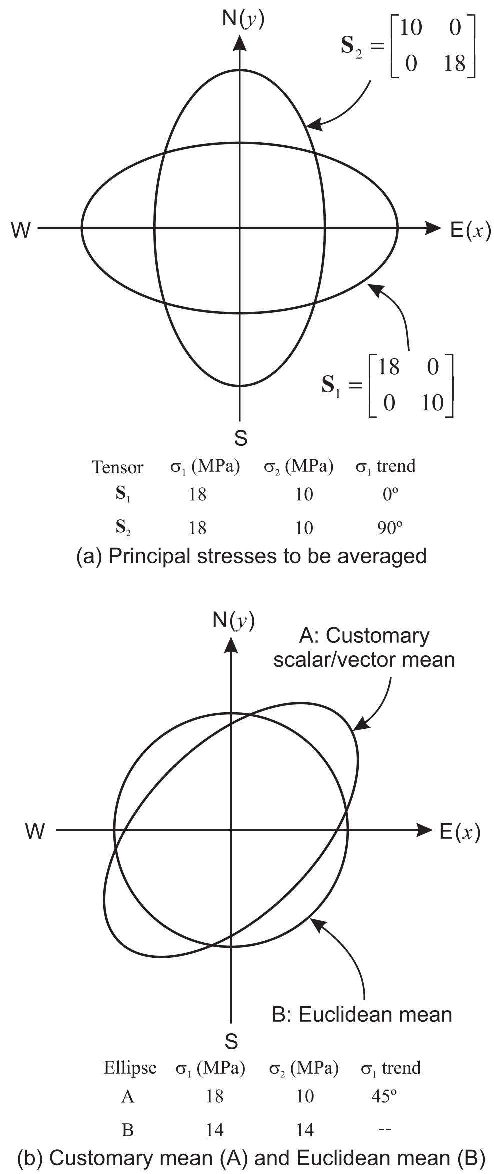


Fig. 2. Demonstration of stress tensor averaging using customary scalar/vector and tensorial approaches (after Dyke et al.[29]).

mean stress tensor.[31–33] This approach first takes a group of n stress measurements in a global x-y-z Cartesian coordinate system, the ith stress tensor $\mathbf{S}_i$ of which is given by

$$\mathbf{S}_i = \begin{bmatrix} \sigma_{x_i} & \tau_{xy_i} & \tau_{xz_i} \\ & \sigma_{y_i} & \tau_{yz_i} \\ \text{symmetric} & & \sigma_{z_i} \end{bmatrix}, \tag{3}$$

where σ and τ are the normal and shear tensor components, respectively. For convenience, we introduce the half-vectorisation function vech(·), which stacks only the lower triangular (i.e. on and below the diagonal) columns of a symmetric matrix.[41 (p. 246)] For example, for the tensor of Eq. (3), we have vector

$$\mathbf{s}_i = \mathrm{vech}(\mathbf{S}_i) = [\sigma_{x_i}\ \ \tau_{yx_i}\ \ \tau_{zx_i}\ \ \sigma_{y_i}\ \ \tau_{zy_i}\ \ \sigma_{z_i}]^T = [\sigma_{x_i}\ \ \tau_{xy_i}\ \ \tau_{xz_i}\ \ \sigma_{y_i}\ \ \tau_{yz_i}\ \ \sigma_{z_i}]^T \tag{4}$$

containing the six distinct components, where $[\cdot]^T$ denotes the matrix transpose. In terms of $\mathbf{s}$ the mean of these six tensor components is then given by

$$\bar{\mathbf{s}} = \mathrm{vech}(\bar{\mathbf{S}}) = [\bar{\sigma}_x\ \ \bar{\tau}_{xy}\ \ \bar{\tau}_{xz}\ \ \bar{\sigma}_y\ \ \bar{\tau}_{yz}\ \ \bar{\sigma}_z]^T, \tag{5}$$

where $\bar{\mathbf{S}}$ is the Euclidean mean stress tensor,[40] i.e.

$$\bar{\mathbf{S}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ & \bar{\sigma}_y & \bar{\tau}_{yx} \\ \text{symmetric} & & \bar{\sigma}_z \end{bmatrix} = \frac{1}{n}\begin{bmatrix} \sum_{i=1}^{n}\sigma_{x_i} & \sum_{i=1}^{n}\tau_{xy_i} & \sum_{i=1}^{n}\tau_{xz_i} \\ & \sum_{i=1}^{n}\sigma_{y_i} & \sum_{i=1}^{n}\tau_{yz_i} \\ \text{symmetric} & & \sum_{i=1}^{n}\sigma_{z_i} \end{bmatrix}. \tag{6}$$

A number of reports exist in which this Euclidean mean has been used as a mean stress tensor.[29,31–33,42,43]

Continuing, the eigenvectors of $\bar{\mathbf{S}}$ (i.e. the direction of the principal components of the mean stress tensor) are then used to define a new Cartesian coordinate system X-Y-Z, and the original stress data transformed into this system. Using the variance function, $\mathrm{var}(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$, and recognising that $\bar{\tau}_{XY} = \bar{\tau}_{YZ} = \bar{\tau}_{ZX} = 0$, the variances of the six distinct tensor components are then calculated by separate processing of each transformed tensor component[31–33,43]:

$$\begin{aligned}\sigma^2_{\mathbf{s}_i} &= [\mathrm{var}(\sigma_X)\ \ \mathrm{var}(\tau_{XY})\ \ \mathrm{var}(\tau_{XZ})\ \ \mathrm{var}(\sigma_Y)\ \ \mathrm{var}(\tau_{YZ})\ \ \mathrm{var}(\sigma_Z)] \\ &= \frac{1}{n-1}\sum_{i=1}^{n}\left[(\sigma_{X_i} - \bar{\sigma}_X)^2\ \ (\tau_{XY_i})^2\ \ (\tau_{XZ_i})^2\ \ (\sigma_{Y_i} - \bar{\sigma}_Y)^2\ \ (\tau_{YZ_i})^2\ \ (\sigma_{Z_i} - \bar{\sigma}_Z)^2\right].\end{aligned} \tag{7}$$

Using the mean and variance of each tensor component calculated in Eqs. (5) and (7), and assuming a normal distribution for each component, a random stress tensor in the X-Y-Z coordinate system is generated by drawing a random value from each of the six independent distributions. Additional random stress tensors are obtained by repeating the sampling procedure.

This approach indeed generates random stress tensors in a tensorial manner, and is theoretically more reasonable than that of separately generating random principal stress magnitudes and orientations from their respective distributions.[7] However, it assumes that each of the six distinct stress tensor components follows an independent univariate distribution, and ignores any statistical correlation between them. We suggest that it is because we routinely refer to the "six independent components" of the stress tensor that we intuitively consider them to be statistically independent elements. However, since the statistical correlation between tensor components is not necessarily zero (as we show below using actual stress data), and thus they may not be statistically independent, we propose a multivariate statistical approach that considers both the variance and covariance of the tensor components.

3. Proposed fully tensorial approach for random stress tensor generation

Random stress tensors can be obtained by applying multivariate statistics to generate random vectors containing the distinct tensor components since we have demonstrated that the variability of stress tensors can be sufficiently and appropriately characterised by their distinct tensor components in a multivariate manner.[39] The theoretical basis of this can be derived from the definition of the symmetric matrix variate distribution, which has been developed to explicitly quantify the variability of symmetric matrix-valued quantities,[34 (p. 71)] and has been revealed to be appropriate for stress variability analysis.[35] Here we use a normal distribution to simply demonstrate this.

By definition, a symmetric matrix $\mathbf{S}_i$ is said to follow a symmetric matrix variate normal distribution with mean $\mathbf{M}$ and covariance matrix $\mathbf{\Omega}$ if and only if the vector $\mathbf{s}_i = \mathrm{vech}(\mathbf{S}_i)$ follows a multivariate normal distribution with mean $\boldsymbol{\mu} = \mathrm{vech}(\mathbf{M})$ and covariance matrix $\mathbf{\Omega} = \mathrm{cov}(\mathbf{s}_i)$, where cov(·) denotes the covariance function.[41 (p. 428)] Matrix variate statistics and multivariate statistics are often used interchangeably since they have the same statistics and probability density function.[36–39] Thus, instead of generating the whole stress tensor, it is convenient to generate a vector containing the distinct tensor components in a multivariate manner and use these components to form a stress tensor.

Since the multivariate normal distribution is the most widely used distribution type, and observations are often seen to be approximately normally distributed,[34] the proposed fully tensorial random vector generation approach employs a multivariate normal statistical model as an example, which requires the mean and covariance matrix of the distinct tensor components of the measured *in situ* stress data as inputs. In addition, this use of the normal distribution allows us to compare the proposed approach to the existing quasi-tensorial approach, since the latter also uses the normal distribution. The multivariate normal model implies that the measured *in situ* stresses and the generated stress tensors are all samples from the same multivariate normal population. For data that follow a multivariate normal distribution, maximum likelihood estimation (MLE) of parameters is recommended[44 (p. 311)] as it gives more robust estimation of the variance and covariance than the unbiased estimation presented in Eq. (7).

After transforming the stress measurements into a global *x-y-z* Cartesian coordinate system, the MLE of the mean and covariance matrix of the six distinct tensor components are

$$\hat{\boldsymbol{\mu}} = \bar{\mathbf{s}} = [\bar{\sigma}_x \;\; \bar{\tau}_{xy} \;\; \bar{\tau}_{xz} \;\; \bar{\sigma}_y \;\; \bar{\tau}_{yz} \;\; \bar{\sigma}_z]^T \tag{8}$$

and

$$\hat{\boldsymbol{\Omega}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{s}_i - \bar{\mathbf{s}})(\mathbf{s}_i - \bar{\mathbf{s}})^T = \begin{bmatrix} \mathrm{var}(\sigma_x) & \mathrm{cov}(\sigma_x,\tau_{xy}) & \mathrm{cov}(\sigma_x,\tau_{xz}) & \mathrm{cov}(\sigma_x,\sigma_y) & \mathrm{cov}(\sigma_x,\tau_{yz}) & \mathrm{cov}(\sigma_x,\sigma_z) \\ & \mathrm{var}(\tau_{xy}) & \mathrm{cov}(\tau_{xy},\tau_{xz}) & \mathrm{cov}(\tau_{xy},\sigma_y) & \mathrm{cov}(\tau_{xy},\tau_{yz}) & \mathrm{cov}(\tau_{xy},\sigma_z) \\ & & \mathrm{var}(\tau_{xz}) & \mathrm{cov}(\tau_{xz},\sigma_y) & \mathrm{cov}(\tau_{xz},\tau_{yz}) & \mathrm{cov}(\tau_{xz},\sigma_z) \\ & & & \mathrm{var}(\sigma_y) & \mathrm{cov}(\sigma_y,\tau_{yz}) & \mathrm{cov}(\sigma_y,\sigma_z) \\ & & & & \mathrm{var}(\tau_{yz}) & \mathrm{cov}(\tau_{yz},\sigma_z) \\ \text{symmetric} & & & & & \mathrm{var}(\sigma_z) \end{bmatrix}, \tag{9}$$

respectively. Here, n in the denominator of Eq. (9) indicates MLE. Finally, multivariate random vector generation using the estimated mean and covariance matrix of Eqs. (8) and (9),[45 (p. 197)] as routinely applied in multivariate statistics, is used to generate random stress tensor components.

A direct way of obtaining random vectors that follow a multivariate normal distribution is to first generate a vector that contains six independent and identically distributed standard normal random numbers – say, $\mathbf{v} = [v_1 \;\; v_2 \;\; v_3 \;\; v_4 \;\; v_5 \;\; v_6]^T$ – and then form the random vector

$$\mathbf{s}_r = \mathbf{L}\mathbf{v} + \hat{\boldsymbol{\mu}} \tag{10}$$

where $\mathbf{L}$ is a 6×6 lower triangular matrix obtained by Cholesky decomposition of the covariance matrix in Eq. (9) such that $\mathbf{L}\mathbf{L}^T = \hat{\boldsymbol{\Omega}}$.[45 (p. 197)] Then, vector $\mathbf{s}_r$ follows a multivariate normal distribution $N_6(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Omega}})$. Repeating this procedure generates a series of random stress data.

Eq. (10) shows that generation of random stress tensors is essentially an application of multivariate random vector generation. Mathematical software packages such as MATLAB[46] and GNU Octave[47] provide the function *mvnrnd* for multivariate normal random vector generation, and in the current application this function directly takes as arguments the mean vector and covariance matrix calculated by Eqs. (8) and (9), respectively.

Comparing the proposed fully tensorial approach to the existing quasi-tensorial one presented in Section 2.2 shows that the existing quasi-tensorial approach is essentially a multivariate method that uses a matrix $\mathbf{L}$ formed from decomposition of a diagonal covariance matrix in which the main diagonal elements are variances of the six distinct tensor components, i.e.

$$\begin{bmatrix} \mathrm{var}(\sigma_X) & 0 & 0 & 0 & 0 & 0 \\ & \mathrm{var}(\tau_{XY}) & 0 & 0 & 0 & 0 \\ & & \mathrm{var}(\tau_{XZ}) & 0 & 0 & 0 \\ & & & \mathrm{var}(\sigma_Y) & 0 & 0 \\ & & & & \mathrm{var}(\tau_{YZ}) & 0 \\ \text{symmetric} & & & & & \mathrm{var}(\sigma_Z) \end{bmatrix} = \mathrm{diag}(\sigma_{\mathbf{s}_i}^2). \tag{11}$$

Here, the $\mathrm{diag}(\cdot)$ function creates a square diagonal matrix with the input elements on the leading diagonal. As the off-diagonal elements, which represent correlations, have the value of zero, the existing quasi-tensorial approach is seen to assume statistical independence of tensor components. The inappropriateness of this is shown next, using actual *in situ* stress data.

4. Application, comparison and discussion

To give a detailed demonstration of the proposed fully tensorial random tensor generation approach and compare it with the existing quasi-tensorial method, here we use both approaches to generate distributions of the tensor components associated with actual stress data.

4.1. Comparison between the existing quasi-tensorial and proposed fully tensorial approaches

17 complete stress tensors obtained at a depth of around 417 m have been extracted from the *in situ* stress measurements made at the AECL's Underground Research Laboratory.[6] The data, transformed into the common coordinate system of x East, y North and z vertically upwards, together with their Euclidean mean are presented in Table 1. Since the existing quasi-tensorial approach only generates random tensors in the coordinate system aligned with the direction of the principal components of the mean stress tensor, in order to compare both approaches we first transform the stress data into this coordinate system. The principal stress directions are the eigenvectors of the Euclidean mean stress tensor, and are found to be

$$\mathbf{R}^T = \begin{bmatrix} 0.1037 & -0.9792 & -0.1743 \\ -0.9913 & -0.1160 & 0.0615 \\ -0.0805 & 0.1664 & -0.9828 \end{bmatrix}, \tag{12}$$

where the three column vectors correspond to the directions of σ_1, σ_2 and σ_3, respectively, referred to the *x-y-z* frame.

The 17 stress data are now transformed into a new *X-Y-Z* Cartesian coordinate system that coincides with the eigenvectors using ordinary stress transformation, i.e.

$$\mathbf{S}_i' = \mathbf{R}\mathbf{S}_i\mathbf{R}^T, \tag{13}$$

where $\mathbf{S}_i'$ denotes a stress tensor in the *X-Y-Z* coordinate system. The mean and covariance matrix of the six distinct components are

Table 1
In situ stress tensor components and their Euclidean mean in the *x-y-z* coordinate system (data from Martin[6]).

Depth (m)	Stress tensor components (MPa)					
	σ_x	τ_{xy}	τ_{xz}	σ_y	τ_{yz}	σ_z
416.55	43.25	4.67	−3.44	32.67	−0.34	15.35
416.57	41.20	6.59	−3.32	31.30	0.46	17.69
416.60	42.92	8.80	−3.97	35.83	2.83	14.57
416.62	45.11	5.42	−4.44	31.59	2.29	18.34
416.68	42.57	4.36	−1.93	28.27	0.85	15.13
416.69	53.78	5.26	−2.26	31.51	3.62	17.61
416.70	26.05	−7.48	−2.57	38.40	1.74	12.35
416.71	28.85	−12.01	−5.65	45.40	6.71	16.29
416.73	30.96	−9.73	−3.86	42.67	0.45	14.56
416.77	23.88	−9.88	−3.70	51.36	1.09	15.19
416.79	34.97	−14.97	−4.51	57.51	1.80	11.74
416.81	27.89	−10.89	−1.60	44.53	−0.24	14.22
417.17	33.78	6.06	−2.19	46.27	0.19	14.59
417.17	33.09	6.35	−5.77	45.00	0.10	18.15
417.17	26.07	4.60	−3.30	42.37	3.14	12.69
417.17	28.18	4.70	−3.89	40.82	3.72	18.25
417.17	29.73	3.00	−4.92	40.55	−0.08	14.22
Euclidean mean	34.84	−0.30	−3.61	40.36	1.67	15.35

$$\hat{\boldsymbol{\mu}}' = [40.52 \quad 0 \quad 0 \quad 35.42 \quad 0 \quad 14.61]^T \text{ MPa} \tag{14}$$

and

$$\hat{\boldsymbol{\Omega}}' = \begin{bmatrix} 74.53 & -38.37 & -11.65 & -56.12 & -7.95 & -9.17 \\ & 45.74 & 9.07 & 32.75 & 0.97 & 5.46 \\ & & 5.79 & 10.52 & 0.43 & 2.01 \\ & & & 78.81 & 10.49 & 9.86 \\ & & & & 3.35 & 0.62 \\ \text{symmetric} & & & & & 3.67 \end{bmatrix} \text{MPa}^2, \tag{15}$$

respectively. The non-zero off-diagonal elements in Eq. (15) indicates correlation between the various distinct stress tensor components, and suggests that assuming these to be zero (i.e. the existing quasi-tensorial approach) is incorrect. The existing quasi-tensorial approach summarised in Section 2.2 uses as the covariance matrix the clearly different diagonal matrix comprising the variances of tensor components, which for these data is

$$\text{diag}(\sigma^2_{s_i}) = \begin{bmatrix} 74.53 & 0 & 0 & 0 & 0 & 0 \\ & 45.74 & 0 & 0 & 0 & 0 \\ & & 5.79 & 0 & 0 & 0 \\ & & & 78.81 & 0 & 0 \\ & & & & 3.35 & 0 \\ \text{symmetric} & & & & & 3.67 \end{bmatrix} \text{MPa}^2. \tag{16}$$

The above calculations are particularly significant when we consider the customary understanding in rock mechanics that the complete stress tensor comprises "six independent components". Statistically, this is only true if the correlations between the stress components are zero.[13 (p. 73)] However, Eq. (9) in general will not lead to correlations between tensor components of zero (as Eq. (15) demonstrates), and treating tensor components as independent entities will introduce errors in applications such as random tensor generation. We therefore suggest using the term "six distinct components" rather than "six independent components" in order to be statistically correct[48 (p. 56)] as well as avoiding misinterpretations.

Using the above statistics of actual *in situ* stress data, we generate large numbers of random tensors using both the existing quasi-tensorial and proposed fully tensorial approaches and compare their differences. Generating 5×10^6 random tensors, using the existing quasi-tensorial approach summarised in Section 2.2 and the proposed fully tensorial approach in Section 3, and using n in place of $(n-1)$ in the denominator of Eq. (7), resulted in a covariance matrix for the existing quasi-tensorial approach of

$$\hat{\boldsymbol{\Omega}}'_e = \begin{bmatrix} 74.58 & -0.01 & 0.01 & 0.01 & -0.01 & -0.01 \\ & 45.70 & 0 & 0 & -0.01 & 0 \\ & & 5.79 & -0.01 & 0 & 0 \\ & & & 78.80 & 0.01 & -0.01 \\ & & & & 3.35 & 0 \\ \text{symmetric} & & & & & 3.67 \end{bmatrix} \text{MPa}^2, \tag{17}$$

and for the proposed fully tensorial approach of

$$\hat{\boldsymbol{\Omega}}'_p = \begin{bmatrix} 74.57 & -38.39 & -11.65 & -56.17 & -7.96 & -9.18 \\ & 45.76 & 9.07 & 32.79 & 0.98 & 5.46 \\ & & 5.79 & 10.52 & 0.43 & 2.01 \\ & & & 78.85 & 10.50 & 9.87 \\ & & & & 3.35 & 0.62 \\ \text{symmetric} & & & & & 3.68 \end{bmatrix} \text{MPa}^2. \tag{18}$$

The almost zero off-diagonal elements in Eq. (17) and the almost identical leading diagonals of Eqs. (15) and (17) demonstrate that the existing quasi-tensorial approach indeed generates statistically independent tensor components the variances of which are those of the measured data. To compare the difference between Eqs. (15) and (18), we introduce a distance measure approach – Euclidean distance, which is commonly used to compare the difference between matrices or vectors of the same dimensions, e.g. Dutilleul[38]. For example the difference between matrices **A** and **B** can be quantified by their Euclidean distance, i.e.

$$d(\mathbf{A}, \mathbf{B}) = \|\mathbf{A} - \mathbf{B}\|_F, \tag{19}$$

where $\|\cdot\|_F$ denotes the Frobenius norm (also called Euclidean norm).[41 (p. 72)] The Euclidean distance between Eqs. (15) and (18) is found to be 0.12 MPa^2, rather than zero, but we believe this small value is a result of bias related to random sampling. We thus conclude that the proposed fully tensorial method generates tensors whose variances and covariances are equivalent to those of the measured data.

The sequence of the distinct tensor components used above is the one shown in Eq. (4). If we change the sequence to put shear components first, followed by normal components to give

$$\mathbf{s}'_i = [\tau_{xy_i} \quad \tau_{xz_i} \quad \tau_{yz_i} \quad \sigma_{x_i} \quad \sigma_{y_i} \quad \sigma_{z_i}]^T, \tag{20}$$

and generate 5×10^6 random tensors using both the existing quasi-tensorial and proposed fully tensorial approaches, then the mean and covariance matrix obtained using the existing quasi-tensorial approach are

$$\hat{\boldsymbol{\mu}}'_e = [0 \quad 0 \quad 0 \quad 40.52 \quad 35.42 \quad 14.61]^T \text{ MPa} \tag{21}$$

and

$$\hat{\boldsymbol{\Omega}}'_e = \begin{bmatrix} 45.73 & -0.01 & -0.01 & 0.03 & -0.01 & 0 \\ & 5.79 & 0 & -0.01 & -0.01 & 0 \\ & & 3.35 & 0 & 0.01 & 0 \\ & & & 74.48 & -0.01 & 0 \\ & & & & 78.83 & -0.01 \\ \text{symmetric} & & & & & 3.68 \end{bmatrix} \text{MPa}^2, \tag{22}$$

respectively, and the mean and covariance matrix obtained using the proposed fully tensorial approach are

$$\hat{\boldsymbol{\mu}}'_p = [0 \quad 0 \quad 0 \quad 40.52 \quad 35.42 \quad 14.61]^T \text{ MPa} \tag{23}$$

and

$$\hat{\boldsymbol{\Omega}}'_p = \begin{bmatrix} 45.79 & 9.08 & 0.98 & -38.42 & 32.83 & 5.47 \\ & 5.79 & 0.43 & -11.66 & 10.53 & 2.01 \\ & & 3.35 & -7.96 & 10.50 & 0.62 \\ & & & 74.61 & -56.20 & -9.18 \\ & & & & 78.89 & 9.86 \\ \text{symmetric} & & & & & 3.67 \end{bmatrix} \text{MPa}^2, \tag{24}$$

respectively. Comparing Eqs. (21) and (23) to Eq. (14), and Eqs. (22) and (24) to Eqs. (16) and (15), respectively, shows that the elements of the mean and covariance matrices are identical (barring random sampling effects), although in a different sequence. This demonstrates that the sequence of stress components has no effect on the statistical properties of the group of stress tensors. Nevertheless, for consistency we use the sequence presented in Eq. (4) throughout the remainder of this work.

Probability density distributions of the six tensor components generated by both approaches are shown in Fig. 3, and are seen to be practically identical. As correlations between tensor components cannot be displayed in Fig. 3, we further plot in Fig. 4 the distributions of the principal stress magnitudes and orientations. The distributions of principal stress magnitudes (Fig. 4a-c) are those of the 5×10^6 generated tensors, while to maintain clarity in the distribution of principal orientations (Fig. 4d-f) a random selection of only 500 generated tensors has been plotted. As a further aid to clarity of visualisation and to assist in comparison, the hemispherical projections in each of Fig. 4d-f have been rotated to place the Euclidean mean direction of the corresponding principal stress at the centre, with the Euclidean mean directions of the other principal stresses located at top, bottom, left and right of each projection. The angular differences between the mean principal directions of the 500 tensors and the population are practically insignificant, as shown in Table 2.

Using the generalised variance we have introduced previously in Gao and Harrison[49,50] – i.e.

$$V_g(\mathbf{S}_i) = |\boldsymbol{\Omega}|, \tag{25}$$

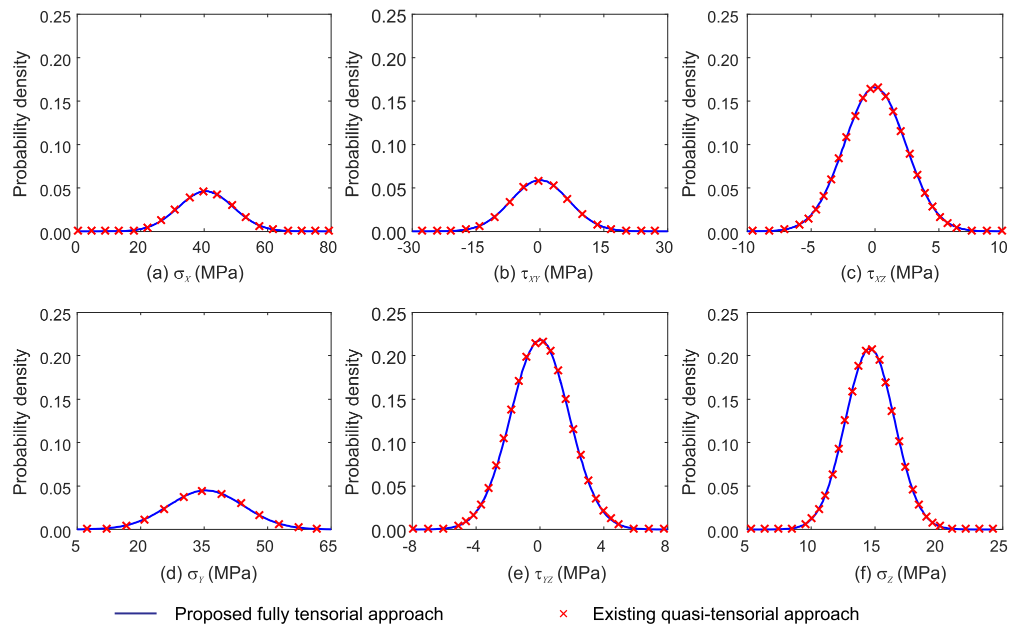


Fig. 3. Probability density distributions of tensor components generated in the *X-Y-Z* coordinate system by both the proposed fully tensorial and existing quasi-tensorial approaches.

where $|\cdot|$ denotes the matrix determinant – to quantify the dispersion of the stress data generated by both approaches (the larger the generalised variance, the more dispersed are the stress data), we find that the proposed fully tensorial approach produces a generalised variance of 6.57×10^5 (i.e. the determinant of Eqs. (15) or (18)), whereas the existing quasi-tensorial approach produces a significantly larger value of 1.91×10^7 (i.e. the determinant of either Eqs. (16) or (17)). In other words, the existing quasi-tensorial approach, which ignores correlation between tensor components, will generate random stresses with greater scatter than will the proposed fully tensorial approach. Indeed, as Fig. 4 shows, the magnitudes of σ_1 and σ_2, and the orientations of σ_2 and σ_3 generated by the proposed fully tensorial approach, appear more concentrated than those associated with the existing quasi-tensorial approach. Note that these determinants have units of (stress)^{12}, and so their magnitudes are dependent on the stress units in use. Continuing, we note that the probability density distributions resulting from the proposed fully tensorial approach and shown in Fig. 5 are generally similar to those illustrated in Fig. 1a, in particular in terms of the distinct overlap of the distributions associated with each principal stress. Finally, a comparison of Fig. 4, Fig. 5 and Eq. (18) suggests that the wide dispersion of σ_1 and σ_2 magnitudes and orientations may be related to the elevated magnitudes of variance and covariance associated with σ_x, σ_y and τ_{xy} (i.e. $\mathrm{var}(\sigma_x)=74.57$, $\mathrm{var}(\sigma_y)=78.85$, $\mathrm{var}(\tau_{xy})=45.76$ and $\mathrm{cov}(\sigma_x,\sigma_y)=-56.17$), as these are more than 1 order of magnitude larger than those associated with σ_z (i.e. $\mathrm{var}(\sigma_z)=3.68$). Further investigation is being conducted to confirm this.

4.2. Transformational consistency of the proposed fully tensorial approach

In the above analysis, random tensors are generated in the coordinate system that coincides with the principal directions of the mean stress tensor. However, for practical application it is important that the approach can be used with any convenient coordinate system, and so here we examine transformational consistency with respect to coordinate systems. Transformational consistency in this context can be interpreted such that after transforming the random tensors generated in different coordinate systems into a common frame, all tensor groups should have the same mean and covariance matrix.

In order to test the transformational consistency of the proposed fully tensorial approach, we first transform the 5×10^6 random tensors generated previously in the *X-Y-Z* principal stress coordinate system into the original *x-y-z* coordinate system (i.e. *x* East, *y* North and *z* vertically upwards) and compute the mean $\hat{\boldsymbol{\mu}}_1$ and covariance $\hat{\boldsymbol{\Omega}}_1$ of these. Additionally, another 5×10^6 random tensors are generated directly in the original *x-y-z* coordinate system using the statistics of the actual *in situ* stress tensors in the *x-y-z* coordinate system shown in Table 1, i.e. the mean

$$\hat{\boldsymbol{\mu}} = [34.84 \quad -0.30 \quad -3.61 \quad 40.36 \quad 1.67 \quad 15.35]^T \ \text{MPa} \tag{26}$$

and the covariance matrix

$$\hat{\boldsymbol{\Omega}} = \begin{bmatrix} 67.59 & 34.96 & 1.74 & -42.09 & 0.11 & 7.01 \\ & 63.61 & 0.72 & -40.24 & -1.75 & 7.86 \\ & & 1.43 & -2.92 & -0.63 & -0.59 \\ & & & 58.29 & 0.38 & -6.85 \\ & & & & 3.33 & 0.63 \\ \text{symmetric} & & & & & 4.13 \end{bmatrix} \text{MPa}^2, \tag{27}$$

and the mean $\hat{\boldsymbol{\mu}}_2$ and covariance $\hat{\boldsymbol{\Omega}}_2$ of these calculated. The mean tensors of the two generated tensor groups are

$$\hat{\boldsymbol{\mu}}_1 = [34.83 \quad -0.31 \quad -3.61 \quad 40.36 \quad 1.67 \quad 15.35]^T \ \text{MPa} \tag{28}$$

and

$$\hat{\boldsymbol{\mu}}_2 = [34.83 \quad -0.30 \quad -3.61 \quad 40.36 \quad 1.67 \quad 15.35]^T \ \text{MPa}, \tag{29}$$

respectively, which are seen to be identical to practical precision and have a Euclidean distance between them of $d(\hat{\boldsymbol{\mu}}_1, \hat{\boldsymbol{\mu}}_2)=0.01$ MPa. The covariance matrices of the two groups are

$$\hat{\boldsymbol{\Omega}}_1 = \begin{bmatrix} 67.60 & 34.97 & 1.74 & -42.09 & 0.11 & 7.01 \\ & 63.67 & 0.72 & -40.24 & -1.76 & 7.86 \\ & & 1.43 & -2.93 & -0.63 & -0.58 \\ & & & 58.28 & 0.38 & -6.85 \\ & & & & 3.33 & 0.63 \\ \text{symmetric} & & & & & 4.13 \end{bmatrix} \text{MPa}^2 \tag{30}$$

and

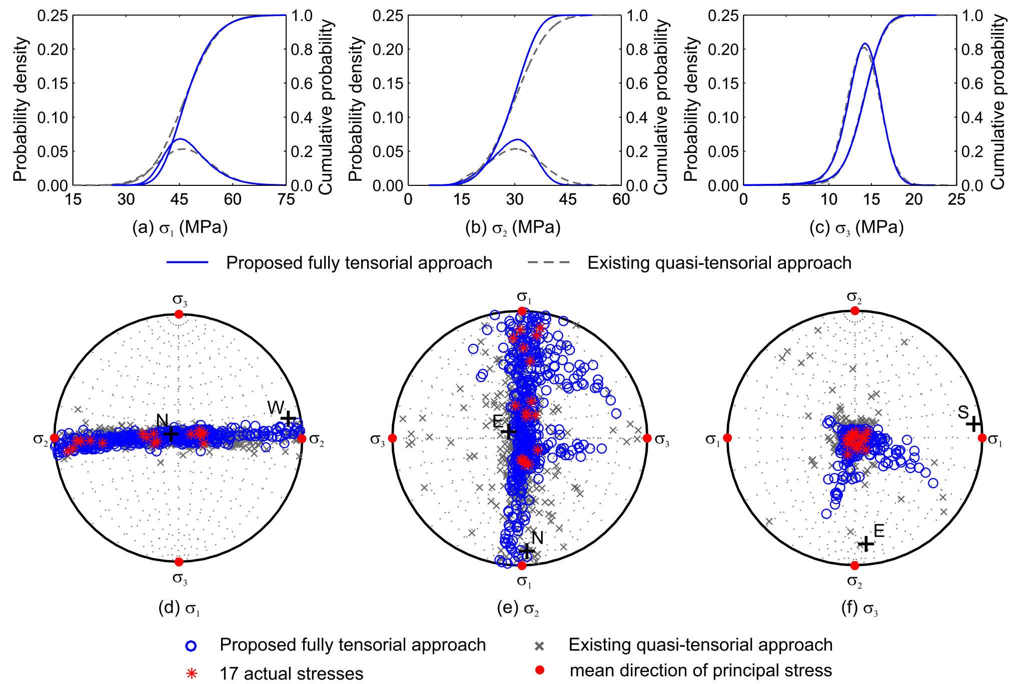


Fig. 4. Distributions of principal stress magnitudes and orientations of random tensors generated by both proposed fully tensorial and existing quasi-tensorial approaches (for clarity the orientation plots show a random selection of only 500 generated tensors, and for improved visualisation and easier comparison each hemispherical projection has been rotated to place the Euclidean mean at the centre of the projection and the other two Euclidean mean principal stress directions at the N-S and E-W positions).

$$\hat{\boldsymbol{\Omega}}_2 = \begin{bmatrix} 67.61 & 34.96 & 1.73 & -42.10 & 0.11 & 7.01 \\ & 63.62 & 0.72 & -40.25 & -1.75 & 7.86 \\ & & 1.43 & -2.93 & -0.63 & -0.59 \\ & & & 58.32 & 0.38 & -6.86 \\ & & & & 3.34 & 0.63 \\ \text{symmetric} & & & & & 4.13 \end{bmatrix} \text{MPa}^2, \tag{31}$$

respectively, with a Euclidean distance between these of $d(\hat{\boldsymbol{\Omega}}_1, \hat{\boldsymbol{\Omega}}_2)$ $=0.07\ \text{MPa}^2$. That the means and covariance matrices are not identical is once again, we believe, an artefact of the random sampling. However, the negligible difference between the transformed tensor groups in terms of both mean and covariance matrix shows the groups to be transformationally invariant. This is supported by the plots in Fig. 6, which show the probability density distributions of the six distinct tensor components of the two groups to be practically identical.

Table 2
Typical angular differences between the mean principal directions of the 500 generated data and the population for both proposed fully tensorial and existing quasi-tensorial approaches (note that these values will change for each group of random tensors).

	σ_1 (deg)	σ_2 (deg)	σ_3 (deg)
Proposed fully tensorial approach	1.2	1.2	0.3
Existing quasi-tensorial approach	1.2	1.1	0.4

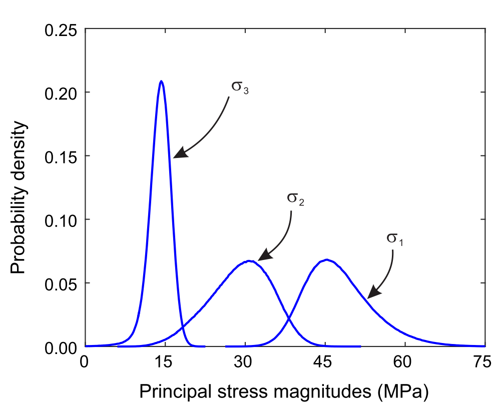


Fig. 5. Probability density distributions of principal stress magnitudes of random tensors generated by the proposed fully tensorial approach.

A large number of additional calculations using different coordinate systems, but not given here due to space constraints, confirm the transformational consistency. This indicates that the proposed fully tensorial approach can generate random stress tensors in any convenient coordinate system, in contrast to the existing quasi-tensorial approach which is limited to generating random tensors in the coordinate system that coincides with the principal directions of the Euclidean mean stress tensor. However, further analytical investigations are needed to identify how the statistics and distributions of the distinct tensor components are related across different coordinate systems.

In the above analyses we have assumed that the distinct tensor components follow a multivariate normal distribution, but it is not yet known what probability distribution is best suited to *in situ* stresses. When information regarding the underlying probability distribution of *in situ* stress tensors becomes available, the methodology presented here can be used but with the appropriate distribution being substituted for the multivariate normal distribution.

5. Conclusions

We have proposed a multivariate approach to generating random stress tensors using the mean and covariance matrix of the distinct tensor components of a sample of stress tensors referred to a common Cartesian coordinate system.

The proposed fully tensorial approach uses the covariance matrix of the distinct tensor components, and thus considers both the variance of and covariance between the components. This is in contrast to the existing quasi-tensorial method, which uses a diagonal matrix comprising the

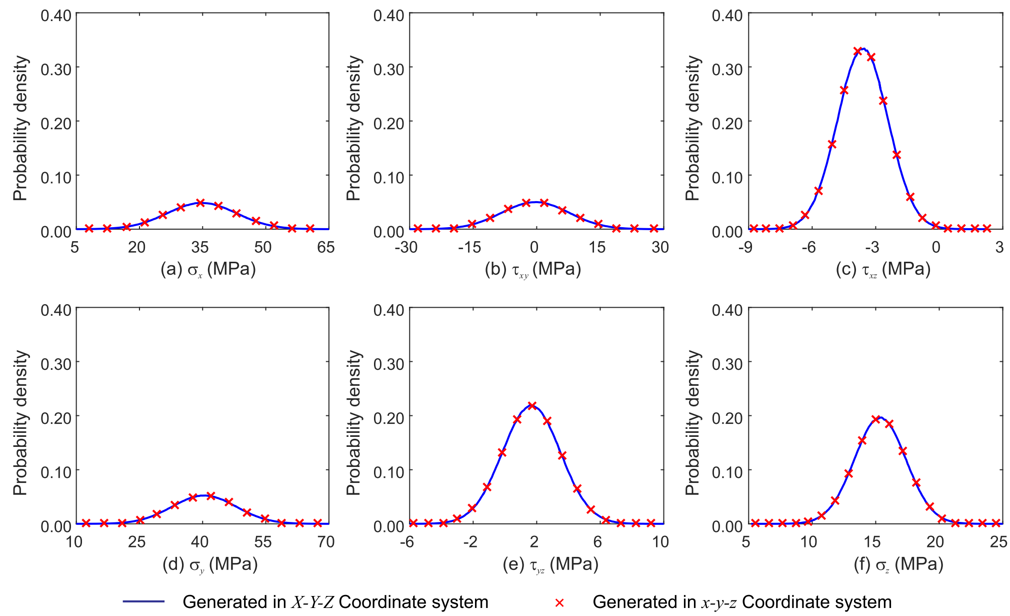


Fig. 6. Probability density distributions of tensor components generated by the proposed fully tensorial approach in both *X-Y-Z* and *x-y-z* coordinate systems, and transformed into a common *x-y-z* coordinate system.

variances of the distinct tensor components as the covariance matrix, and thereby ignores any statistical dependence between the tensor components. To explicitly recognise the potential for statistical dependence between components, in order to be consistent with mathematical nomenclature, and to avoid misinterpretations, we recommend referring to the tensor components as "six distinct components", rather than the heretofore customary "six independent components".

We have compared the proposed fully tensorial and existing quasi-tensorial approaches by generating random tensors in a Cartesian coordinate system (*X-Y-Z*) aligned with the principal directions of the Euclidean mean of the sample tensors. Our results suggest application of the existing quasi-tensorial approach (which ignores covariance) leads to greater scatter of generated tensors than does application of the proposed fully tensorial approach (which includes covariance).

We have demonstrated the transformational consistency of the proposed fully tensorial approach by generating random stress tensors in different coordinate systems, and comparing the mean, covariance matrix and distributions of the stress tensor components in the context of a common coordinate system. The transformational consistency of the proposed fully tensorial approach allows generation of random tensors in any convenient coordinate system, which is in contrast to the existing quasi-tensorial approach that only permits generation of random tensors in the coordinate system corresponding to the principal directions of the Euclidean mean stress tensor of the source sample of tensors.

The proposed fully tensorial approach provides a method that will assist with probabilistic or reliability-based analyses of rock engineering structures when considering the inherent variability of *in situ* stress.

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