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# Mean and dispersion of stress tensors using Euclidean and Riemannian approaches

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ABSTRACT

Stress is central to many aspects of rock mechanics, and in the analysis of *in situ* stress measurement data the calculation of the mean value and an assessment of dispersion are important for statistical characterisation. Currently, stress magnitude and orientation are processed separately in such analyses. This effectively decomposes the second-order stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, and calculation of mean and dispersion of stress data on the basis of these decomposed components, which violates the tensorial nature of stress, may either yield biased results or be difficult to conduct. Here, by introducing tensorial techniques, we examine two calculation approaches for the mean and dispersion for stress tensors – based on Euclidean and Riemannian geometries – and discuss their similarities, differences and potential applicability in engineering practice. We compare the two approaches using stress tensor superposition and interpolation, and the analysis of actual *in situ* stress data. The results indicate that Euclidean and Riemannian mean tensors are in general not equal, with the disparity increasing as stress tensor dispersion increases. Both Euclidean and Riemannian approaches are shown to be capable of characterising stress dispersion, although Euclidean dispersion is scale dependent and has units of stress whereas Riemannian dispersion is a scale independent unitless number. Finally, a paradox is revealed in that despite stress tensors being Riemannian entities, it is Euclidean mean stress that is the more meaningful for engineering applications.


## 1. Introduction

Stress is central to many aspects of rock mechanics, and in the analysis of *in situ* stress measurement data the calculation of the mean value and an assessment of dispersion are important for statistical characterisation.[8,12,13,32–36] Currently, stress magnitude and orientation are customarily processed separately in such analyses (Fig. 1).[1–8,32–34,36] This effectively decomposes the second-order stress tensor into scalar (principal stress magnitudes) and vector (principal stress orientations) components, and calculation of mean and dispersion of stress data on the basis of these decomposed components, which violates the tensorial nature of stress, may either yield biased results or be difficult to conduct[9,10,13,37 (p54)]. As noted elsewhere,[11] '*Since stress is a tensor with six independent components, calculating the mean, standard deviation and confidence intervals of the measured stresses cannot be carried out using the same statistical techniques developed for scalar quantities*'. As an alternative to the separate analysis of principal stress magnitude and orientation, several researchers in the field of rock mechanics have calculated the mean stress tensor based on tensors referred to a common Cartesian coordinate system.[11–14,35] Although these contributions essentially introduced a tensorial approach, they did so in an empirical setting. A result of this is that, to date, there seems to have been no mathematically rigorous proposal from the rock mechanics community for calculating such summary statistics for groups of stress tensors as the mean and dispersion. In particular, the calculation of the dispersion of a group of stress tensors obtained from a stress measurement campaign seems not to have been conducted in the rock mechanics field. Here, continuing the analysis of stress tensors referred to a common Cartesian coordinate system, and considering tensors as single entities, we introduce approaches based on Euclidean and Riemannian geometry to calculate their mean and dispersion.

As an early tensorial application example in rock mechanics, Hyett et al.[12] demonstrated that the mean of $n$ stress tensors should be found by firstly transforming the individual tensors to a common Cartesian coordinate system (say, $x$–$y$–$z$), and then calculating the mean of each tensor component:

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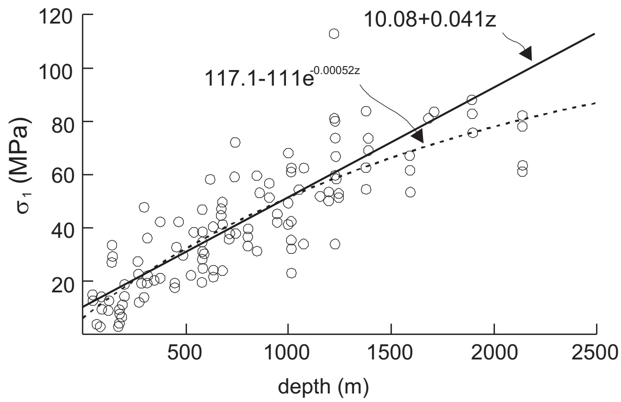
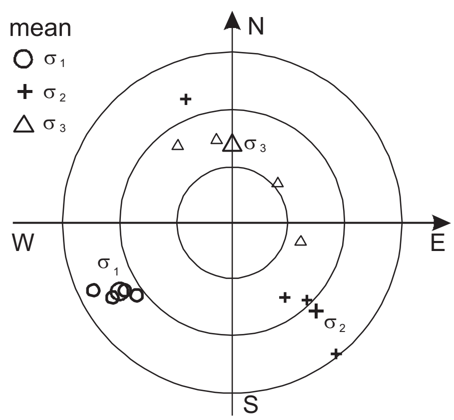



(a) Least squares regression of stress magnitude




(b) Directional statistics applied to principal stress orientation

**Fig. 1.** Example separate analyses of stress magnitude and orientation.[1] (a) Least squares regression of stress magnitude. (b) Directional statistics applied to principal stress orientation.

$$\bar{\mathbf{S}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i = \begin{bmatrix} \bar{\sigma}_x & \bar{\tau}_{xy} & \bar{\tau}_{xz} \\ & \bar{\sigma}_y & \bar{\tau}_{yx} \\ \text{symmetric} & & \bar{\sigma}_z \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n}\sigma_{x_i} & \frac{1}{n}\sum_{i=1}^{n}\tau_{xy_i} & \frac{1}{n}\sum_{i=1}^{n}\tau_{xz_i} \\ & \frac{1}{n}\sum_{i=1}^{n}\sigma_{y_i} & \frac{1}{n}\sum_{i=1}^{n}\tau_{yz_i} \\ \text{symmetric} & & \frac{1}{n}\sum_{i=1}^{n}\sigma_{z_i} \end{bmatrix}. \tag{1}$$

Here $\bar{\mathbf{S}}$ denotes the mean stress tensor, $\mathbf{S}_i$ represents a particular stress tensor, $\sigma$ and $\tau$ are the normal and shear tensor components, respectively, and $\bar{\sigma}$ and $\bar{\tau}$ denote the corresponding mean tensor components. This approach was subsequently followed by others.[11,13,14,35]

Based on Eq. (1), several researchers[11,13,35,39] suggested how the variance of stress tensors might be calculated. After obtaining the mean stress tensor, a new coordinate system is established that coincides with the principal directions of the mean tensor (say, $X$–$Y$–$Z$), and all the original stress tensors transformed into this new coordinate system. Using the fundamental definition of variance, i.e. $\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)$, and recognising that $\bar{\tau}_{XY}=\bar{\tau}_{YZ}=\bar{\tau}_{ZX}=0$, the variance tensor is then calculated as

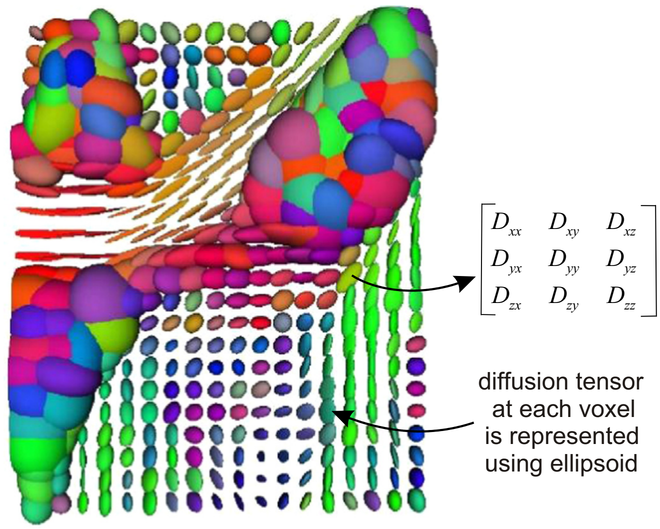



**Fig. 2.** Diffusion tensor at each voxel in Diffusion Tensor Imaging.[17]

$$\sigma^2_{\mathbf{S}} = \frac{1}{n-1}\begin{bmatrix} \sum_{i=1}^{n}\left(\sigma_{X_i}-\bar{\sigma}_X\right)^2 & \sum_{i=1}^{n}\left(\tau_{XY_i}\right)^2 & \sum_{i=1}^{n}\left(\tau_{XZ_i}\right)^2 \\ & \sum_{i=1}^{n}\left(\sigma_{Y_i}-\bar{\sigma}_Y\right)^2 & \sum_{i=1}^{n}\left(\tau_{YZ_i}\right)^2 \\ \text{symmetric} & & \sum_{i=1}^{n}\left(\sigma_{Z_i}-\bar{\sigma}_Z\right)^2 \end{bmatrix}. \tag{2}$$

However, this only gives the dispersion of each tensor component, rather than a scalar value indicating the overall variability of the group of tensors. In other words, comparison of the dispersions of different groups of tensors is still difficult to conduct with this approach.

As we show below, these rock mechanics tensorial applications are essentially a Euclidean approach. However, it is now known that symmetric positive definite (SPD) matrices such as stress tensors with positive principal stresses do not live in Euclidean space, but in curved spaces known as Riemannian manifolds (see for example, Chapter 6 of Ref. 15 and Chapter 19 of Ref. 16). Statistical analysis of SPD matrices on Riemannian manifolds has been recently developed for use in Magnetic Resonance Imaging (MRI) applications in medicine (Fig. 2). MRI can be used to detect diffusion of water molecules through biological tissues, and analysis of this can reveal microscopic details about tissue architecture (either normal or in a diseased state). As diffusion can be characterised by an SPD matrix called the "diffusion tensor", it has been necessary to develop tensorial approaches to aid diagnosis.[17–19]

In this paper, and following on from earlier work of ours,[20] we focus on the illustration and comparison of Euclidean and Riemannian approaches to calculating the mean and dispersion of stress tensors, and their potential applicability in engineering practice. The underlying stochastic model is one that simultaneously includes all stress tensor components (when referred to a common Cartesian coordinate system), rather than one that processes principal stress magnitudes and orientations separately. Since these calculations are based on distance measures, we first give a simple comparison of Euclidean and Riemannian distances, and indicate their significance in the calculation of mean tensors. Following this we introduce both Euclidean and Riemannian mean and dispersion functions, and present tensor superposition techniques in both Euclidean and Riemannian spaces. We move on to compare Euclidean and Riemannian approaches through stress tensor superposition and interpolation, and the analysis of actual and perturbed *in situ* stress data. We conclude by examining the differences between the two approaches and their respective

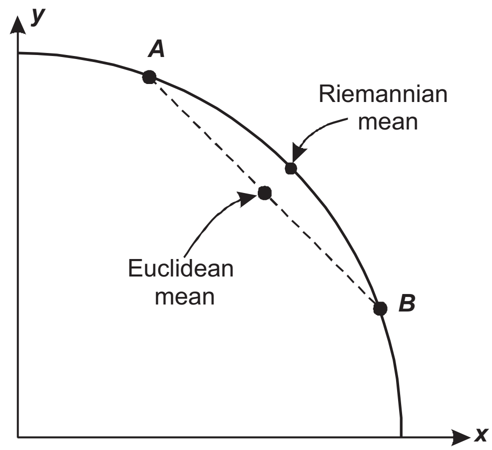



**Fig. 3.** Distance between two points and their Euclidean and Riemannian means.

applicability in engineering practice, and highlight aspects of these analyses that may warrant further attention by the rock mechanics community.

## 2. Euclidean and Riemannian distance measures

Two commonly used geometries are Euclidean and Riemannian. A characteristic of Euclidean geometry is that the distance between two points is the length of the straight line connecting them, whereas in Riemannian geometry the distance between two points is the length, on the surface of the Riemannian manifold, of the minimum-length curve joining them. Such curves are known as geodesics. These concepts are illustrated in Fig. 3. The Euclidean distance between points *A* and *B* is the length of the line section connecting them, with the arithmetic mean of *A* and *B* being their midpoint. However, as *A* and *B* lie on a circle (a Riemannian manifold), we see that their mean should lie at the midpoint of geodesic *AB*, and the distance between them must be calculated along the circle, i.e. in a Riemannian sense. A concrete example is given by locations on the Earth's surface. If we consider the Canadian cities Toronto and Vancouver, with WGS coordinates (43.7°, −79.4°) and (49.25°, −123.1°) respectively, and assuming a spherical Earth of radius 6371 km, the Riemannian distance between them is about 3354 km and the Euclidean distance about 3316 km. Although these differ by only about 1%, the Euclidean midpoint is almost 220 km below the Earth's surface. This simple example demonstrates that a fundamental difference between these two geometries is the applicable distance measures. Mathematically, Riemannian geometry is more general, with Euclidean geometry being the special case of zero curvature. [21]

The mean and dispersion, being moments,[38] can be calculated on the basis of distance measures. Taking the mean as an example, and using the Fréchet mean function[19] as the fundamental definition of this, we have

$$E(x_i;\, i \in 1:\, n) = \arg\ \min_y \left( \frac{1}{n} \sum_{i=1}^{n} d^2(y, x_i) \right), \tag{3}$$

showing that the mean is the point $y$ that minimises the expectation of the square distance between $y$ and each $x_i$, where $d(y, x_i)$ is the distance between $y$ and $x_i$. As is shown below, it is this dependence on a distance measure that leads to different Euclidean and Riemannian statistics.

## 3. Mean of stress tensors

### 3.1. Euclidean mean stress tensor

To derive a Euclidean mean stress tensor, we start with the derivation of a scalar mean. Since the Euclidean distance between two scalars is their difference, Eq. (3) becomes

$$E(x_i;\, i \in 1:\, n) = \arg\ \min_y \left( \frac{1}{n} \sum_{i=1}^{n} (y - x_i)^2 \right). \tag{4}$$

The mean is the value of $y$ that minimises $\sum_{i=1}^{n} (y - x_i)^2$. By differentiation,

$$\frac{d}{dy}\left( \sum (y - x_i)^2 \right) = 2 \sum (y - x_i) = 2\left( ny - \sum x_i \right) = 0, \tag{5}$$

and rearranging this gives the familiar scalar Euclidean mean:

$$E(x_i;\, i \in 1:\, n) = y = \frac{1}{n} \sum_{i=1}^{n} x_i. \tag{6}$$

For stress tensors, Eq. (3) is written as

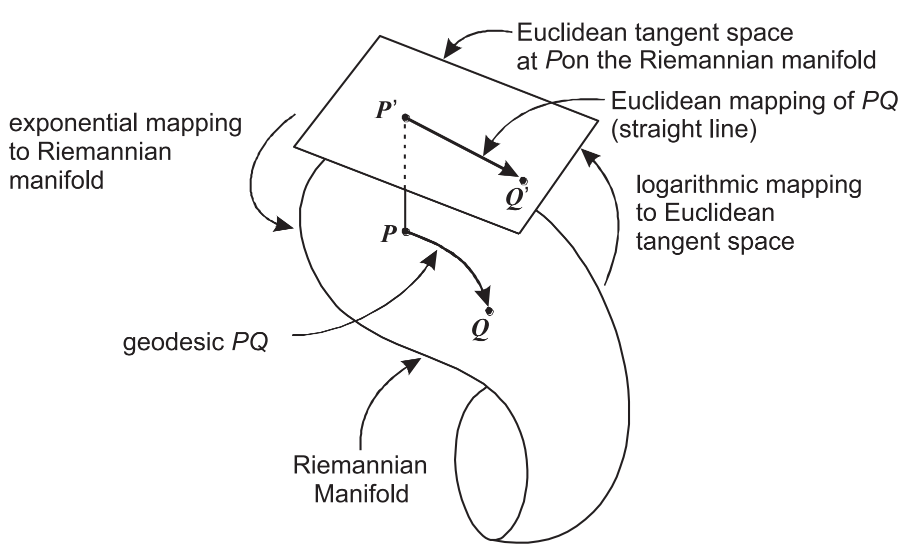



**Fig. 4.** Mapping between a Riemannian manifold and a Euclidean tangent space.

$$E(\mathbf{S}_i;\ i \in 1{:}\ n) = \arg\min_{\mathbf{Y}}\left(\frac{1}{n}\sum_{i=1}^{n} d^2(\mathbf{Y}, \mathbf{S}_i)\right), \tag{7}$$

where $\mathbf{S}_i$ are the known stress tensors and $\mathbf{Y}$ is the mean tensor. The Euclidean distance between tensors $\mathbf{S}_1$ and $\mathbf{S}_2$ is[22]

$$d^2(\mathbf{S}_1, \mathbf{S}_2) = \sum_{i=1}^{3}\sum_{j=1}^{3}\left(S_{ij|1} - S_{ij|2}\right)^2 = \|\mathbf{S}_1 - \mathbf{S}_2\|_F^2, \tag{8}$$

where $\|\cdot\|_F$ denotes the Frobenius norm (also called Euclidean norm or Hilbert-Schmidt norm) [23]. For a $3 \times 3$ tensor $\mathbf{S}$ the Frobenius norm is given by [23]

$$\|\mathbf{S}\|_F = \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}^2} = \sqrt{\mathrm{tr}\left(\mathbf{S}\cdot\mathbf{S}^T\right)} = \sqrt{\mathrm{tr}\left(\mathbf{S}^2\right)}, \tag{9}$$

where $T$ and $\mathrm{tr}(\cdot)$ denote matrix transpose and trace, respectively.

To be meaningful, the Euclidean distance needs to be transformationally invariant (i.e. independent of the coordinate system). To confirm that it is, we use Eq. (9) to consider the case when the stress tensors are subject to some transformation represented by the matrix $\mathbf{R}$, i.e. $\mathbf{S}' = \mathbf{R}\cdot\mathbf{S}\cdot\mathbf{R}^T$. Thus

$$\begin{aligned} d^2(\mathbf{S}'_1, \mathbf{S}'_2) &= \|\mathbf{S}'_1 - \mathbf{S}'_2\|_F^2 = \|\mathbf{R}\mathbf{S}_1\mathbf{R}^T - \mathbf{R}\mathbf{S}_2\mathbf{R}^T\|_F^2 = \|\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)\mathbf{R}^T\|_F^2 \\ &= \mathrm{tr}\left(\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)\mathbf{R}^T\cdot\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)\mathbf{R}^T\right) \\ &= \mathrm{tr}\left(\mathbf{R}(\mathbf{S}_1 - \mathbf{S}_2)^2\mathbf{R}^T\right) = \mathrm{tr}\left((\mathbf{S}_1 - \mathbf{S}_2)^2\right) = \|\mathbf{S}_1 - \mathbf{S}_2\|_F^2 \\ &= d^2(\mathbf{S}_1, \mathbf{S}_2). \end{aligned} \tag{10}$$

For the mean, using Eq. (8), Eq. (7) becomes

$$E(\mathbf{S}_i;\ i \in 1{:}\ n) = \arg\min_{\mathbf{Y}}\left(\frac{1}{n}\sum_{i=1}^{n}\|\mathbf{Y} - \mathbf{S}_i\|_F^2\right), \tag{11}$$

from which, in the same manner as Eq. (6), the Euclidean mean stress tensor is obtained as

$$\bar{\mathbf{S}}_E = E(\mathbf{S}_i;\ i \in 1{:}\ n) = \mathbf{Y} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i. \tag{12}$$

This is transformationally consistent, in that the mean of transformed tensors is equal to the transformed mean:

$$\bar{\mathbf{S}}'_E = \frac{1}{n}\sum_{i=1}^{n}\mathbf{R}\mathbf{S}_i\mathbf{R}^T = \mathbf{R}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i\right)\mathbf{R}^T = \mathbf{R}\cdot\bar{\mathbf{S}}_E\cdot\mathbf{R}^T. \tag{13}$$

These results show that it is the Euclidean mean stress tensor that has been calculated previously by those authors who proposed using a common Cartesian coordinate system.[11–14,35]

By applying a scale factor $k$ ($k > 0$) to the individual stress tensors $\mathbf{S}_i$, we see that the Euclidean mean is scale dependent:

$$\bar{\mathbf{S}}'_E = \frac{1}{n}\sum_{i=1}^{n}k\mathbf{S}_i = k\frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i = k\bar{\mathbf{S}}_E. \tag{14}$$

### 3.2. Riemannian mean stress tensor

The Riemannian mean of SPD matrices has been studied by researchers in several areas,[15-19,24-26] since it is known that SPD matrices live in curved Riemannian spaces.[15,16] As the Fréchet mean function, Eq. (3), shows, calculation of the Riemannian mean requires a Riemannian distance. The fundamental approach to obtaining Riemannian distances is first mapping SPD matrices from the Riemannian manifold on to a Euclidean tangent space at

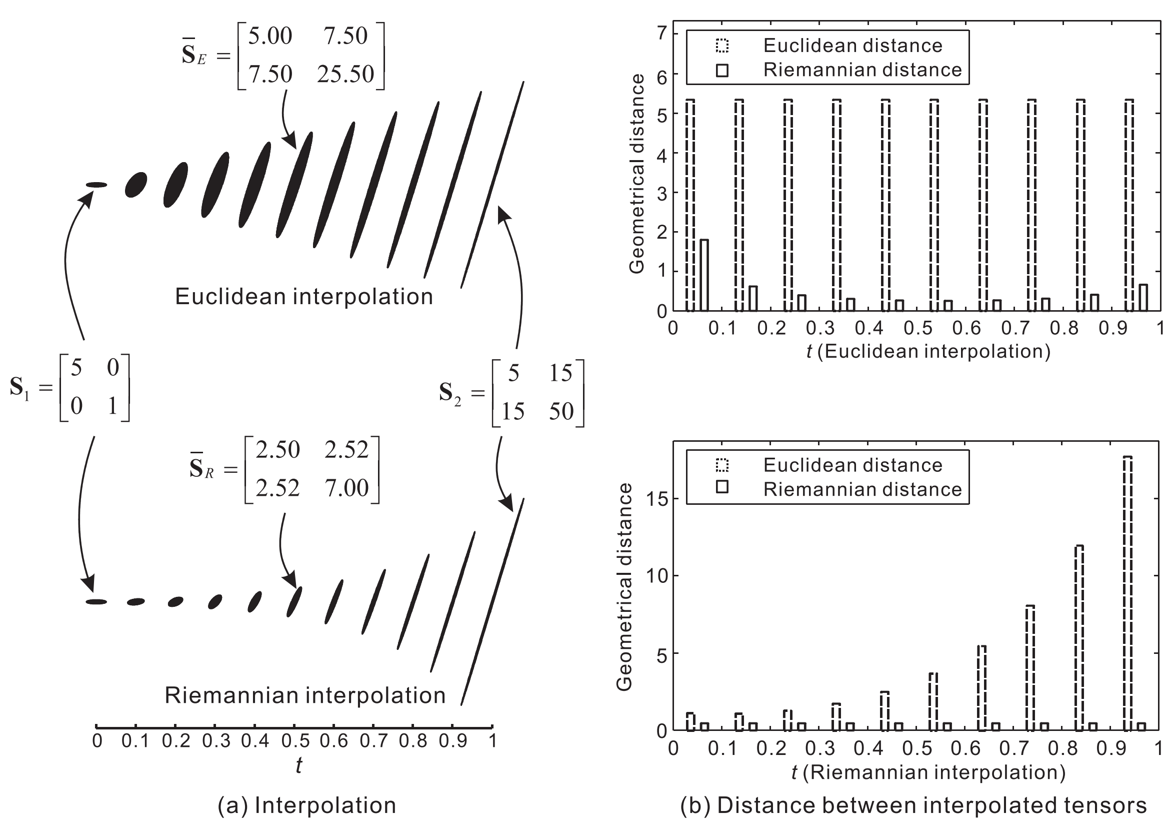



**Fig. 5.** Euclidean and Riemannian interpolations and the distances between interpolated tensors.

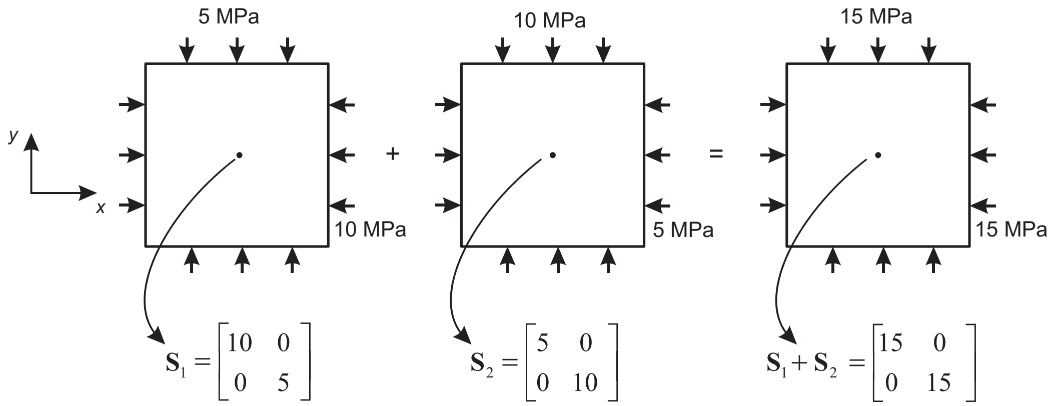



**Fig. 6.** Physical interpretation of tensor superposition.

a specific point on the Riemannian manifold (Fig. 4), and following this by distance calculation using Euclidean concepts.[17,21] Here we briefly introduce two widely used Riemannian distance functions.

Pennec et al.[26] used the affine-invariant distance function

$$d^2(\mathbf{S}_1, \mathbf{S}_2) = \mathrm{tr}\left(\log^2\left(\left(\sqrt{\mathbf{S}_1}\right)^{-1}\mathbf{S}_2\left(\sqrt{\mathbf{S}_1}\right)^{-1}\right)\right) = \left\| \log\left(\left(\sqrt{\mathbf{S}_1}\right)^{-1}\mathbf{S}_2\left(\sqrt{\mathbf{S}_1}\right)^{-1}\right)\right\|_F^2 \quad (15)$$

to calculate the distance between two SPD matrices, where $\log(\cdot)$ represents the matrix version of the logarithm, and $\sqrt{\mathbf{S}_1}\cdot\sqrt{\mathbf{S}_1} = \mathbf{S}_1$. No analytical solution exists for the Riemannian mean using Eqs. (7) and (15), and so a numerical technique, using a gradient descent algorithm, has been proposed:[26]

$$\mathbf{Y}_{t+1} = \sqrt{\mathbf{Y}_t}\,\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log\left(\left(\sqrt{\mathbf{Y}_t}\right)^{-1}\mathbf{S}_i\left(\sqrt{\mathbf{Y}_t}\right)^{-1}\right)\right)\sqrt{\mathbf{Y}_t}. \quad (16)$$

Here, $\exp(\cdot)$ denotes the matrix version of the exponential function. Upon convergence, $\mathbf{Y}$ is the Riemannian mean.

As Eqs. (15) and (16) show, the matrix logarithm and matrix exponential are central to Riemannian computation. These functions are defined[27–29] as

$$\log(\mathbf{S}) = \mathbf{V}\cdot\ln(\mathbf{\Lambda})\cdot\mathbf{V}^T \text{ and } \exp(\mathbf{S}) = \mathbf{V}\cdot\exp(\mathbf{\Lambda})\cdot\mathbf{V}^T, \quad (17)$$

where $\mathbf{\Lambda}$ and $\mathbf{V}$ are the diagonal matrix of the eigenvalues and the orthogonal matrix of the corresponding eigenvectors, respectively, with the functions $\ln(\cdot)$ and $\exp(\cdot)$ operating on each diagonal element of $\mathbf{\Lambda}$. Mathematical software packages such as MATLAB and Octave[30] provide the functions *logm* and *expm* for matrix logarithm and matrix exponential, respectively.

Using Eq. (16) to calculate the mean is usually computationally expensive, and to overcome this the linearized Log-Euclidean distance has been proposed.[17,31] This can be written as

$$d^2(\mathbf{S}_1, \mathbf{S}_2) = \mathrm{tr}\left(\left(\log\mathbf{S}_1 - \log\mathbf{S}_2\right)^2\right) = \|\log\mathbf{S}_1 - \log\mathbf{S}_2\|_F^2 \quad (18)$$

The Log-Euclidean distance requires significantly less computation than does the affine-invariant distance of Eq. (15), and providing the linearization introduces sufficiently small errors the two yield similar (and potentially, identical) results.[17] For simplicity, only the Log-Euclidean distance is used in the remainder of this work. As with the Euclidean distance, the Log-Euclidean distance is transformationally invariant:

**Table 1**
Stress tensor components (data from Ref. 3).

| Depth (m) | Stress tensor components (MPa) | | | | | |
|---|---|---|---|---|---|---|
| | $\sigma_x$ | $\tau_{xy}$ | $\tau_{xz}$ | $\sigma_y$ | $\tau_{yz}$ | $\sigma_z$ |
| 416.55 | 43.25 | 4.67 | −3.44 | 32.67 | −0.34 | 15.35 |
| 416.57 | 41.20 | 6.59 | −3.32 | 31.30 | 0.46 | 17.69 |
| 416.60 | 42.92 | 8.80 | −3.97 | 35.83 | 2.83 | 14.57 |
| 416.62 | 45.11 | 5.42 | −4.44 | 31.59 | 2.29 | 18.34 |
| 416.68 | 42.57 | 4.36 | −1.93 | 28.27 | 0.85 | 15.13 |
| 416.69 | 53.78 | 5.26 | −2.26 | 31.51 | 3.62 | 17.61 |
| 416.70 | 26.05 | −7.48 | −2.57 | 38.40 | 1.74 | 12.35 |
| 416.71 | 28.85 | −12.01 | −5.65 | 45.40 | 6.71 | 16.29 |
| 416.73 | 30.96 | −9.73 | −3.86 | 42.67 | 0.45 | 14.56 |
| 416.77 | 23.88 | −9.88 | −3.70 | 51.36 | 1.09 | 15.19 |
| 416.79 | 34.97 | −14.97 | −4.51 | 57.51 | 1.80 | 11.74 |
| 416.81 | 27.89 | −10.89 | −1.60 | 44.53 | −0.24 | 14.22 |
| 417.17 | 33.78 | 6.06 | −2.19 | 46.27 | 0.19 | 14.59 |
| 417.17 | 33.09 | 6.35 | −5.77 | 45.00 | 0.10 | 18.15 |
| 417.17 | 26.07 | 4.60 | −3.30 | 42.37 | 3.14 | 12.69 |
| 417.17 | 28.18 | 4.70 | −3.89 | 40.82 | 3.72 | 18.25 |
| 417.17 | 29.73 | 3.00 | −4.92 | 40.55 | −0.08 | 14.22 |

$$\begin{aligned} d^2(\mathbf{S}_1', \mathbf{S}_2') &= \|\log\mathbf{S}_1' - \log\mathbf{S}_2'\|_F^2 = \left\|\log\left(\mathbf{R}\mathbf{S}_1\mathbf{R}^T\right) - \log\left(\mathbf{R}\mathbf{S}_2\mathbf{R}^T\right)\right\|_F^2 \\ &= \|\mathbf{R}\cdot\log\mathbf{S}_1\cdot\mathbf{R}^T - \mathbf{R}\cdot\log\mathbf{S}_2\cdot\mathbf{R}^T\|_F^2 = \left\|\mathbf{R}\cdot(\log\mathbf{S}_1 - \log\mathbf{S}_2)\cdot\mathbf{R}^T\right\|_F^2 \\ &= \mathrm{tr}\left(\mathbf{R}(\log\mathbf{S}_1 - \log\mathbf{S}_2)\mathbf{R}^T\cdot\mathbf{R}(\log\mathbf{S}_1 - \log\mathbf{S}_2)\mathbf{R}^T\right) \\ &= \mathrm{tr}\left(\mathbf{R}(\log\mathbf{S}_1 - \log\mathbf{S}_2)^2\,\mathbf{R}^T\right) = \mathrm{tr}\left(\left(\log\mathbf{S}_1 - \log\mathbf{S}_2\right)^2\right) \\ &= \|\log\mathbf{S}_1 - \log\mathbf{S}_2\|_F^2 = d^2(\mathbf{S}_1, \mathbf{S}_2). \end{aligned} \quad (19)$$

Employing Eq. (7), the Riemannian mean is thus[17,31]

$$\bar{\mathbf{S}}_R = E(\mathbf{S}_i;\, i \in 1{:}\, n) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log\mathbf{S}_i\right). \quad (20)$$

This approach first converts tensors into their Euclidean tangent space using the matrix logarithm, then calculates the mean of the transformed tensors in a Euclidean sense and finally obtains the Riemannian mean by transforming the Euclidean mean back to the Riemannian space using matrix exponential.

As with the Euclidean mean, the Riemannian mean is both transformationally consistent and scale dependent:

**Table 2**
Euclidean and Riemannian means of stress data.

| | $\sigma_x$ | $\tau_{xy}$ | $\tau_{xz}$ | $\sigma_y$ | $\tau_{yz}$ | $\sigma_z$ |
|---|---|---|---|---|---|---|
| Euclidean mean (MPa) | 34.84 | −0.30 | −3.61 | 40.36 | 1.67 | 15.35 |
| Riemannian mean (MPa) | 33.01 | −0.37 | −3.65 | 38.74 | 1.64 | 15.14 |

**Table 3**
Euclidean and Riemannian dispersion of stress data.

| | Original data | Scaled data ($k$=2) |
|---|---|---|
| Euclidean dispersion (MPa) | 16.84 | 33.68 |
| Riemannian dispersion | 0.48 | 0.48 |

**Table 4**
Normal and shear stress component perturbation ranges applied to the Riemannian mean of Table 1.

| Perturbation ranges | Normal stress | Shear stress |
|---|---|---|
| Data set 1 | [−2,2] | [−1,1] |
| Data set 2 | [−4,4] | [−2,2] |

**Table 5**
Perturbed stress Data Set 1.

| | Stress tensor components (MPa) | | | | | |
|---|---|---|---|---|---|---|
| | $\sigma_x$ | $\tau_{xy}$ | $\tau_{xz}$ | $\sigma_y$ | $\tau_{yz}$ | $\sigma_z$ |
| | 31.93 | 0.07 | −4.28 | 37.10 | 2.50 | 14.96 |
| | 34.28 | 0.21 | −2.83 | 40.05 | 0.74 | 14.16 |
| | 33.44 | −0.27 | −2.98 | 38.42 | 2.15 | 16.94 |
| | 34.95 | 0.27 | −2.73 | 38.97 | 1.68 | 15.03 |
| | 34.85 | −0.91 | −3.92 | 36.91 | 2.44 | 13.45 |
| | 34.86 | 0.08 | −3.45 | 38.27 | 1.00 | 13.55 |
| | 33.82 | −0.44 | −3.50 | 38.73 | 0.82 | 15.50 |
| | 34.44 | −0.93 | −3.92 | 37.24 | 2.60 | 13.77 |
| | 32.90 | 0.13 | −3.63 | 39.43 | 1.24 | 15.01 |
| | 35.00 | −0.54 | −2.93 | 39.66 | 1.08 | 16.40 |
| | 34.75 | 0.34 | −4.33 | 37.77 | 1.33 | 16.04 |
| | 34.90 | −0.60 | −2.84 | 37.57 | 2.54 | 16.27 |
| | 32.02 | −0.09 | −2.72 | 37.94 | 2.02 | 13.19 |
| | 31.07 | −0.05 | −4.53 | 40.58 | 2.54 | 15.94 |
| | 34.16 | 0.57 | −4.00 | 38.48 | 1.36 | 15.71 |
| | 31.35 | 0.19 | −3.40 | 37.16 | 0.68 | 16.97 |
| | 33.05 | −0.51 | −3.87 | 37.50 | 2.61 | 16.91 |
| | 34.82 | −1.02 | −3.14 | 39.97 | 1.36 | 15.48 |
| | 31.44 | 0.36 | −4.35 | 40.63 | 1.17 | 13.72 |
| | 34.77 | −0.96 | −3.67 | 37.06 | 1.18 | 15.32 |
| Euclidean mean | 33.64 | −0.20 | −3.55 | 38.47 | 1.65 | 15.22 |
| Riemannian mean | 33.60 | −0.20 | −3.56 | 38.44 | 1.66 | 15.15 |

$$\bar{\mathbf{S}}'_R = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log(\mathbf{R}\mathbf{S}_i\mathbf{R}^T)\right) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}(\mathbf{R}\cdot\log\mathbf{S}_i\cdot\mathbf{R}^T)\right)$$
$$= \exp\left(\mathbf{R}\cdot\frac{1}{n}\sum_{i=1}^{n}\log\mathbf{S}_i\cdot\mathbf{R}^T\right) = \mathbf{R}\cdot\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log\mathbf{S}_i\right)\cdot\mathbf{R}^T$$
$$= \mathbf{R}\cdot\bar{\mathbf{S}}_R\cdot\mathbf{R}^T, \quad (21)$$

**Table 6**
Perturbed stress Data Set 2.

| | Stress tensor components (MPa) | | | | | |
|---|---|---|---|---|---|---|
| | $\sigma_x$ | $\tau_{xy}$ | $\tau_{xz}$ | $\sigma_y$ | $\tau_{yz}$ | $\sigma_z$ |
| | 30.33 | 1.30 | −4.35 | 38.02 | 0.89 | 13.21 |
| | 34.79 | 0.54 | −2.46 | 35.26 | 1.90 | 16.80 |
| | 32.77 | −1.35 | −3.50 | 38.31 | 2.16 | 14.59 |
| | 32.94 | −1.58 | −5.01 | 35.52 | 0.86 | 13.57 |
| | 33.33 | −1.12 | −5.14 | 42.01 | 1.06 | 15.26 |
| | 31.59 | 1.58 | −1.95 | 40.28 | 3.49 | 11.64 |
| | 32.32 | 1.44 | −4.53 | 38.39 | 3.27 | 14.57 |
| | 33.11 | −0.74 | −4.13 | 38.35 | 0.03 | 18.55 |
| | 36.88 | −1.24 | −5.33 | 35.15 | 2.51 | 14.89 |
| | 32.54 | −2.32 | −3.21 | 41.26 | 3.43 | 17.56 |
| | 36.50 | 1.62 | −3.46 | 41.50 | 0.80 | 13.78 |
| | 29.69 | −0.77 | −3.71 | 35.29 | 3.53 | 12.50 |
| | 30.65 | −0.20 | −1.78 | 41.84 | 2.65 | 13.56 |
| | 30.78 | −1.17 | −4.00 | 37.08 | 0.59 | 12.69 |
| | 36.48 | 0.00 | −3.48 | 41.32 | 1.44 | 18.15 |
| | 32.19 | −0.92 | −5.38 | 36.11 | 1.43 | 13.29 |
| | 32.71 | −1.88 | −2.45 | 41.55 | 2.77 | 18.15 |
| | 30.32 | −0.45 | −2.83 | 41.83 | −0.01 | 18.68 |
| | 29.52 | −2.29 | −3.48 | 37.75 | 0.87 | 17.87 |
| | 34.17 | −0.09 | −4.70 | 35.84 | 2.57 | 12.27 |
| Euclidean mean | 32.68 | −0.48 | −3.74 | 38.63 | 1.81 | 15.08 |
| Riemannian mean | 32.56 | −0.50 | −3.77 | 38.50 | 1.84 | 14.85 |

**Table 7**
Euclidean and Riemannian dispersions of perturbed data sets.

| | Data Set 1 | Data Set 2 |
|---|---|---|
| Euclidean dispersion (MPa) | 2.67 | 5.04 |
| Riemannian dispersion | 0.12 | 0.22 |

**Table 8**
Difference between two means for each perturbed data set.

| Distance between two means | Data Set 1 | Data Set 2 |
|---|---|---|
| Euclidean distance | 0.088 | 0.298 |
| Riemannian distance | 0.005 | 0.018 |

$$\bar{\mathbf{S}}'_R = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log(k\mathbf{S}_i)\right) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}(\log(k\mathbf{I}) + \log\mathbf{S}_i)\right)$$
$$= \exp\left(\log(k\mathbf{I}) + \frac{1}{n}\sum_{i=1}^{n}\log\mathbf{S}_i\right) = k\exp\left(\frac{1}{n}\sum_{i=1}^{n}\log\mathbf{S}_i\right)$$
$$= k\bar{\mathbf{S}}_R, \quad (22)$$

where $\mathbf{I}$ is the identity matrix with the same dimensions as the stress tensor.

## 4. Stress tensor dispersion

The variability of scalars may be characterised in terms of standard deviation. Here, we consider stress tensor dispersion, i.e. the square root of the second central moment, as an analogous measure. The reliance of second moment on distance means that Euclidean dispersion is

$$D_E(\mathbf{S}_i;\ i \in 1:n) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\|\mathbf{S}_i - \bar{\mathbf{S}}_E\|_F^2}, \tag{23}$$

and Riemannian dispersion is

$$D_R(\mathbf{S}_i;\ i \in 1:n) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\|\log \mathbf{S}_i - \log \bar{\mathbf{S}}_R\|_F^2}. \tag{24}$$

Because of the transformational consistency of both means and the transformational invariance of both distance functions, it is straightforward to show (in the same way as Eqs. (10) and (19)) that these dispersions are transformationally invariant.

Scale dependency of these dispersions may also be examined. For Euclidean dispersion we have

$$\begin{aligned} D'_E(\mathbf{S}_i;\ i \in 1:n) &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|k\mathbf{S}_i - k\bar{\mathbf{S}}_E\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\mathrm{tr}\left(k^2\left(\mathbf{S}_i - \bar{\mathbf{S}}_E\right)^2\right)} \\ &= k\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\mathrm{tr}\left(\left(\mathbf{S}_i - \bar{\mathbf{S}}_E\right)^2\right)} \\ &= k\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\mathbf{S}_i - \bar{\mathbf{S}}_E\right\|_F^2} = kD_E, \end{aligned} \tag{25}$$

while for the Riemannian dispersion, we obtain

$$\begin{aligned} D'_R(\mathbf{S}_i;\ i \in 1:n) &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log(k\mathbf{S}_i) - \log\left(k\bar{\mathbf{S}}_R\right)\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log(k\mathbf{I}\cdot\mathbf{S}_i) - \log\left(k\mathbf{I}\cdot\bar{\mathbf{S}}_R\right)\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log(k\mathbf{I}) + \log \mathbf{S}_i - \log(k\mathbf{I}) - \log \bar{\mathbf{S}}_R\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log \mathbf{S}_i - \log \bar{\mathbf{S}}_R\right\|_F^2} = D_R. \end{aligned} \tag{26}$$

These results show that data scaling propagates into Euclidean dispersion, but not into Riemannian dispersion. This implies Riemannian dispersion is a scale independent intrinsic measure of data variability, in a manner similar to the Mahalanobis distance or correlation coefficient, and suggests it may be valuable for comparison of stress variability.

## 5. Statistics of stress tensor superposition

Superposition of stress tensors is commonplace in mechanics, being used to investigate sequential application of loads and decomposition into spherical and deviatoric components, amongst others. Here we consider the statistics associated with the particular case of a constant SPD matrix **C** being added to or subtracted from each stress tensor.

For the Euclidean approach, the mean of the superposed tensors is

$$\bar{\mathbf{S}}'_E = E\left(\mathbf{S}'_i;\ i \in 1:n\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{S}_i \pm \mathbf{C}\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{S}_i \pm \mathbf{C} = \bar{\mathbf{S}}_E \pm \mathbf{C} \tag{27}$$

and the dispersion is

$$\begin{aligned} D'_E(\mathbf{S}'_i;\ i \in 1:n) &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|(\mathbf{S}_i \pm \mathbf{C}) - \left(\bar{\mathbf{S}}_E \pm \mathbf{C}\right)\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|(\mathbf{S}_i \pm \mathbf{C}) - \left(\bar{\mathbf{S}}_E \pm \mathbf{C}\right)\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\mathbf{S}_i - \bar{\mathbf{S}}_E\right\|_F^2} = D_E. \end{aligned} \tag{28}$$

As expected, addition or subtraction of **C** propagates through to the mean, and dispersion is unaffected.

Riemannian tensor superposition is only defined in terms of mapping into the Euclidean tangent space,[21] with the addition or subtraction of two tensors thus being given by

$$\mathbf{S}' = \exp(\log \mathbf{S}_1 \pm \log \mathbf{S}_2). \tag{29}$$

Hence, the Riemannian mean of the superposed tensors is

$$\begin{aligned} \bar{\mathbf{S}}'_R = E\left(\mathbf{S}'_i;\ i \in 1:n\right) &= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log \mathbf{S}'_i\right) \\ &= \exp\left(\frac{1}{n}\sum_{i=1}^{n}\left(\log \mathbf{S}_i \pm \log \mathbf{C}\right)\right) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log \mathbf{S}_i \pm \log \mathbf{C}\right) \\ &= \exp\left(\log \bar{\mathbf{S}}_R \pm \log \mathbf{C}\right), \end{aligned} \tag{30}$$

and the dispersion is

$$\begin{aligned} D'_R(\mathbf{S}'_i;\ i \in 1:n) &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log \mathbf{S}'_i - \log \bar{\mathbf{S}}'_R\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log \mathbf{S}_i \pm \log \mathbf{C} - \left(\log \bar{\mathbf{S}}_R \pm \log \mathbf{C}\right)\right\|_F^2} \\ &= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left\|\log \mathbf{S}_i - \log \bar{\mathbf{S}}_R\right\|_F^2} = D_R. \end{aligned} \tag{31}$$

These results show the superposed mean to be the Riemannian superposition of the original mean and the constant, and the superposed dispersion to be equal to the original dispersion. Taking into account the nature of Riemannian space, these results are as expected. In the next section, we use different examples to apply both the Euclidean and Riemannian approaches and compare their differences.

## 6. Euclidean and Riemannian analysis of stress – application, comparison and discussion

It seems that none of Euclidean dispersion, Riemannian mean and Riemannian dispersion have been used in rock mechanics studies of stress, and so here we present some elementary analyses to examine the similarities and differences between Euclidean and Riemannian statistics, and their potential applicability in engineering practice. We begin with example calculations of superposition and interpolation of 2D stress tensors, and follow this with calculations using 3D stress tensors obtained as part of an actual stress measurement campaign.

As Eqs. (12) and (20) show, the Euclidean and Riemannian mean values are the arithmetic and geometric means,[15] respectively. The use of the matrix logarithm in Eq. (20) indicates why Riemannian mean is only defined for SPD matrices, and although this restriction has the potential to be a serious drawback to Riemannian analysis, as *in situ* stress tensors are generally SPD matrices (i.e. positive principal stress components when using the geomechanics convention of compression positive) in practice it may prove to be irrelevant.

### 6.1. Superposition and interpolation of 2D stress tensors

Both interpolation and superposition are strongly related to calculation of the mean value. From Eqs. (12) and (20) we see that the first step in obtaining the Euclidean and Riemannian means is to perform summation of stress tensors. In the case of two tensors, this is equivalent to superposition. Similarly, the mean of two tensors is equivalent to the interpolated mid-point, with other interpolated values being considered weighted means (Fig. 5).[17].

#### 6.1.1. Stress tensor superposition

As the Euclidean mean (Eq. (12)) and Riemannian mean (Eq. (20)) functions show, stress tensor summation (or superposition) forms the basis of mean calculation. Here we use a straightforward case to demonstrate the physical meaning of stress superposition, and further show the appropriateness of Euclidean and Riemannian mean functions in practice. Fig. 6 shows two 2D stress states applied to a homogeneous, isotropic and elastic plate, and the final stress state resulting from their customary superposition:

$$\mathbf{S}_1 + \mathbf{S}_2 = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}. \tag{32}$$

Reference to Eq. (32) confirms that such customary superposition is a Euclidean analysis.

The Riemannian addition of these two stress tensors is given by Eq. (29), with the result

$$\exp(\log \mathbf{S}_1 + \log \mathbf{S}_2) = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}. \tag{33}$$

This is clearly different from the physical reality with which we are familiar – i.e. Fig. 6 and Eq. (32), indicating that Euclidean superposition (i.e. addition) may be more meaningful than Riemannian superposition. This suggests that the Euclidean mean will be more appropriate than Riemannian mean in engineering applications, although it is not known whether there may be circumstances in which the opposite is true.

#### 6.1.2. Interpolation between stress tensors

In general, and as a consideration of geometry suggests (cf. Fig. 3), Euclidean and Riemannian mean tensors will be different. We investigate the magnitude of this difference using linear interpolation between two stress tensors, as interpolation produces a weighted mean.[17]

Euclidean interpolation is given by

$$\mathbf{S}(t) = (1 - t)\mathbf{S}_1 + t\mathbf{S}_2, \tag{34}$$

and Riemannian interpolation by

$$\mathbf{S}(t) = \exp\left[ (1 - t)\log \mathbf{S}_1 + t \log \mathbf{S}_2 \right], \tag{35}$$

where $0 \le t \le 1$, and $\mathbf{S}_1$ and $\mathbf{S}_2$ are the two tensors being interpolated between. The Riemannian interpolation again follows the Riemannian procedure of first transforming the tensors into Euclidean tangent space, then performing Euclidean interpolation, and finally transforming the interpolation results back into the Riemannian space. With $t = 0.5$ (i.e. the mid-point) it is clear that Eqs. (34) and (35) are equivalent to Eqs. (12) and (20) respectively. The interpolation is illustrated in Fig. 5a using 2D tensors, with each tensor represented by an ellipse whose semi-axes denote the magnitude and orientation of the principal values. Fig. 5b shows the Euclidean and Riemannian distance between adjacent intermediate stress tensors for both forms of interpolation.

Interpretation of these results is perplexing. Firstly, the Euclidean mean seems inherently rational, and is what other authors[11–14,35] would calculate, but this may be because of our familiarity with Euclidean concepts for stress analysis. However, as stress tensors live in Riemannian space, strictly it is the Riemannian mean that is correct despite it seeming physically incomprehensible. We see also that the two mean tensors differ significantly, but this may be a result of the large Euclidean and Riemannian distances between $\mathbf{S}_1$ and $\mathbf{S}_2$ (i.e. 53.39 and 4.52, respectively); Fig. 3 demonstrates this effect, in that the Euclidean and Riemannian mean would approach each other as the distance between *A* and *B* reduces. In effect, over small distances the Euclidean and Riemannian geometries are practically interchangeable (as large-scale maps of small areas of the Earth's surface demonstrate).

Secondly, it is clear that the two interpolation sequences differ significantly. Although the Euclidean interpolation results appear reasonable, they nevertheless seem 'awkward' in the region $0 \le t \le 0.2$. Certainly, the Riemannian results seem to have an overall smoother transition. However – and as noted above – the Riemannian mean value seems somehow 'incorrect', which leaves the meaning of the Riemannian interpolation questionable. It is currently unknown how these contrasting results should be reconciled, but – as we noted above – in this case the distance between $\mathbf{S}_1$ and $\mathbf{S}_2$ is large, and again we speculate that the difference between the two interpolation sequences is a result of this.

### 6.2. Analysis of in situ stress data

To apply these results to actual stress measurement data, 17 complete stress tensors obtained at a depth of around 417 m have been extracted from the 99 *in situ* stress measurements made at the AECL's Underground Research Laboratory.[3] All of these tensors have positive principal stresses and thus are suitable for both Euclidean and Riemannian analyses. The data are presented in Table 1, transformed into the common coordinate system of *x* East, *y* North and *z* vertically upwards.

The Euclidean and Riemannian mean tensors are shown in Table 2, and in this case they are similar. This similarity is, we believe, due to the small dispersion ($D_R = 0.48$) of these tensors, as under such conditions the Euclidean tangent plane is a good approximation to the local Riemannian manifold. At this point we speculate that as the dispersion increases, the discrepancy between these two means also increases; we investigate this point below.

Table 3 shows both Euclidean and Riemannian dispersions, calculated using Eqs. (23) and (24). Note that Euclidean dispersion has units of stress, whereas Riemannian dispersion is a unitless number. The results also confirm Eqs. (25) and (26), in that Euclidean dispersion is scale dependent whereas Riemannian dispersion is scale independent.

To further investigate the application of Eqs. (23) and (24) to characterise stress variability, we generate two other data sets, each comprising 20 tensors, by introducing random perturbations to the Riemannian mean stress components of Table 2. The perturbations are drawn from uniform distributions, the ranges of which are shown in Table 4, and added to the components of Table 2 to give the data shown in Tables 5 and 6.

As the perturbations of the Data set 1 are less than those of Data set 2, we would expect the dispersion associated with the former to be less than the latter. The Euclidean and Riemannian dispersions shown in Table 7 confirm this, indicating that both dispersions are capable of characterising stress variability. Further investigations are needed to identify how the overall dispersions are related to (i) the dispersions of the principal stress magnitudes and the principal stress directions, and (ii) the scalar dispersions of the individual stress components.

We speculated above that the difference between the Euclidean and Riemannian mean tensors for Data set 1 should be less than for Data set 2, since the former set has smaller dispersion than the latter. Table 8 shows both the Euclidean and Riemannian distance between the two mean tensors for each data set. Although the magnitudes of these distances are much smaller than those shown in Fig. 5b, they nevertheless support our speculation.

## 7. Conclusions

We have introduced approaches to calculating the mean and dispersion of stress tensors based on distance measures in both Euclidean and Riemannian spaces. Both approaches provide a tensorial technique for processing of stress tensors, and are thus more appropriate than the informal methods customarily used in rock mechanics. The similarities and differences between Euclidean and Riemannian statistics have been discussed, and their potential applicability in engineering practice noted.

Elementary analyses involving stress tensor superposition and interpolation show that Euclidean and Riemannian approaches may lead to different results. When performing stress tensor superposition and interpolating between two stress tensors, the Euclidean approach yields rational results whereas the Riemannian approach does not. We speculate that the discrepancy between the results increases as the distance between the stress tensors increases.

We have used Euclidean and Riemannian approaches to calculate the mean and dispersion of a group of real *in situ* stress data. The two approaches yield similar results for the mean tensor, and both are able to characterise stress dispersion. However, Euclidean dispersion is scale dependent and has units of stress whereas Riemannian dispersion is a unitless number and scale independent. These features lead us to recommend Riemannian dispersion for characterising stress tensor variability.

Perturbing these data to generate synthetic data sets possessing different variability shows that as the dispersion increases, so the difference between the Euclidean and Riemannian mean tensors increases. This supports our speculation that the discrepancy between the two approaches increases as the distance between stress tensors increases.

We close with the observation that although Euclidean and Riemannian approaches both give cogent results for the calculation of dispersion, with regard to calculation of a mean tensor there seems to be a paradox: stress tensors live within a Riemannian space, but it is only a Euclidean analysis that gives meaningful results. We leave this paradox as an open problem.

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